MODERN CLASSICAL PHYSICS

KIP S. THORNE
and
ROGER D. BLANDFORD

Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics notation, $F_j = T_{jk}\Sigma_k = Pg_{jk}An_k = PAn_j$; i.e. it is a normal force with magnitude equal to the fluid pressure P times the surface area A. This is what it should be.

The stress tensor plays a central role in the *Newtonian law of momentum conservation* because (by definition) the force acting across a surface is the same as the rate of flow of momentum, per unit area, across the surface; i.e., *the stress tensor is the flux of momentum.*

Consider the three-dimensional region of space $\mathcal{V}_3$ used above in formulating the integral laws of charge and particle conservation (1.29). The total momentum in $\mathcal{V}_3$ is $\int_{\mathcal{V}_3} \boldsymbol{\mathcal{G}} dV$, where $\boldsymbol{\mathcal{G}}$ is the momentum density. This changes as a result of momentum flowing into and out of $\mathcal{V}_3$. The net rate at which momentum flows outward is the integral of the stress tensor over the surface $\partial\mathcal{V}_3$ of $\mathcal{V}_3$. Therefore, by analogy with charge and particle conservation (1.29), *the integral law of momentum conservation says*

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}_3} \boldsymbol{\mathcal{G}} dV + \int_{\partial\mathcal{V}_3} \mathbf{T} \cdot d\boldsymbol{\Sigma} = 0} . \tag{1.35}$$

By pulling the time derivative inside the volume integral (where it becomes a partial derivative) and applying the vectorial version of Gauss's law to the surface integral, we obtain $\int_{\mathcal{V}_3} (\partial\boldsymbol{\mathcal{G}}/\partial t + \boldsymbol{\nabla} \cdot \mathbf{T}) dV = 0$. This can be true for all choices of $\mathcal{V}_3$ only if the integrand vanishes:

$$\boxed{\frac{\partial \boldsymbol{\mathcal{G}}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} = 0 \, , \quad \text{i.e.} \quad \frac{\partial \mathcal{G}_j}{\partial t} + T_{jk;k} = 0} . \tag{1.36}$$

This is *the differential law of momentum conservation.* It has the standard form for any local conservation law: the time derivative of the density of some quantity (here momentum), plus the divergence of the flux of that quantity (here the momentum flux is the stress tensor), is zero. We shall make extensive use of this Newtonian law of momentum conservation in Part IV (elasticity theory), Part V (fluid mechanics) and Part VI (plasma physics).

****

EXERCISES

Exercise 1.13 ***Example: Equations of Motion for a Perfect Fluid*

(a) Consider a perfect fluid with density ρ, pressure P, and velocity $\mathbf{v}$ that vary in time and space. Explain why the fluid's momentum density is $\boldsymbol{\mathcal{G}} = \rho\mathbf{v}$, and explain why its momentum flux (stress tensor) is

$$\boxed{\mathbf{T} = P\mathbf{g} + \rho\mathbf{v} \otimes \mathbf{v} \, , \quad \text{or, in slot-naming index notation,} \quad T_{ij} = Pg_{ij} + \rho v_i v_j} . \tag{1.37a}$$

(b) Explain why the law of mass conservation for this fluid is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{v}) = 0. \tag{1.37b}$$