

# Momentum flow diagrams for just-rigid static structures

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**Abstract.** Flow diagrams are a powerful tool for visualizing the current distribution in networks of well defined channels. They can often be interpreted at a single glance. The procedure is common for substance, energy, heat and electric currents. However, it can also be applied to the flow of momentum. We show by means of examples from the statics of plane trusses that such diagrams are easy to draw and to interpret.

(Some figures in this article appear in colour in the electronic version; see [www.iop.org](http://www.iop.org))

## 1. Introduction

Whenever a body exerts a force on another body the momenta of both bodies changes. The increase of the momentum of one of them is equal to the decrease of the momentum of the other. This fact suggests that one can interpret a force as the current strength of a momentum current or momentum flow. However, such an interpretation also presupposes that a local current density can be defined. Max Planck (1908) showed that this is indeed possible. Although this interpretation could facilitate substantially the treatment of mechanical processes, it is used only occasionally in the advanced literature, see, for example, Landau and Lifshitz (1959). Yet, just for beginners, it could be most beneficial (Herrmann 2000, Borer *et al* 2000). Apparently, the secular tradition of Newtonian action-at-a-distance mechanics has prevented a change in the language and conception of momentum.

Just as for the flow of other extensive quantities, the flow of momentum can be represented by stream line diagrams. In this paper, we shall consider a special case of such stream line pictures: situations in which momentum currents flow in well defined ‘channels’. We are familiar with such situations in the case of the flow of other quantities: water, which is constrained to pipes or to river-beds, or electric currents flowing in copper wires. The mechanical version of such discrete flow networks is the mechanics of trusses.

In a previous paper (Herrmann and Schmid 1984) it was shown how the momentum flow in plane trusses can be calculated and the method was applied to a simple example.

The power of a graphical tool depends to a high degree on how easy it is to produce qualitative pictures. To illustrate this point let us consider the familiar situation of a field line picture of an electric field. Every physics student learns rules for drawing such pictures qualitatively: field lines begin and end on electric charges, they enter perpendicularly into the surface of an electric conductor, they never cross, etc. A picture drawn by only using these rules is sufficient for many purposes. Moreover, the ability to draw qualitative field line pictures goes together with the ability to interpret such pictures, whatever the method by which they had

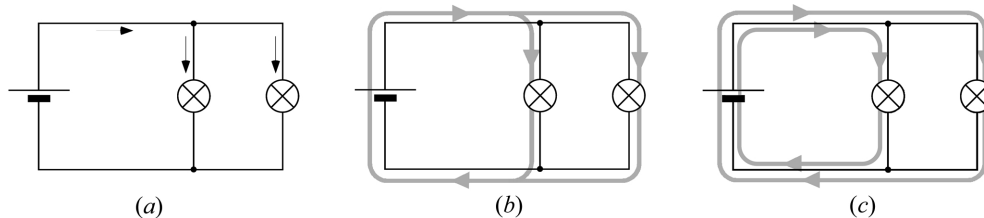
been created. The situation is similar with regard to momentum flow diagrams. Momentum flow diagrams are valuable, because it is possible to interpret them at a glance.

It is the purpose of this paper, to compile rules which enable the reader to draw momentum flow diagrams without analysing the system quantitatively. We begin, in section 2, by considering the graphical methods for representing flow distributions in discrete channels. In section 3 it will be shown how momentum flow distributions in plane trusses can be calculated and interpreted. Section 4 contains a collection of rules for drawing momentum stream lines. In section 5 we shall treat an example in detail. The online version of the paper contains the flow diagrams of 20 more examples. Another advantage of the online version of the paper is that the flow diagrams are in colour.

## 2. The graphical representation

In the momentum flow representation of mechanics, the interpretation of some basic equations of mechanics reveals that they are analogous to the interpretation of equations from electricity (Herrmann and Schmid 1985a). Just as in electricity, we speak about the quantity  $Q$  as if it is a kind of substance, and about the quantity  $I$  as a measure of how much of this substance flows in a given time through a chosen cross section, momentum can be pictured as a ‘stuff’, and  $F$  as a measure of how much of this stuff crosses a reference surface per unit of time. Thus, the equations  $I = dQ/dt$  and  $F = dp/dt$  correspond to each other. In the same way, Kirchhoff’s junction rule for electrical currents  $\sum I_k = 0$  corresponds to a junction rule for momentum currents  $\sum F_k = 0$ . The traditional interpretation of the last equation is that it is an expression of the equilibrium of forces.

Whereas the characteristic extensive quantity of electricity, namely  $Q$ , is a scalar, that of mechanics, i.e. the momentum  $p$ , is a vector. This difference has an effect on the possibilities of a graphical representation of the respective flow distributions. Whereas for the representation of the flow of  $Q$  a single stream line diagram is needed (just as in the case of other scalar quantities, like mass or energy), the representation of the momentum flow requires three stream line pictures: one for the flow of the  $x$  component, one for the flow of the  $y$  component and one for the flow of the  $z$  component of the momentum (Herrmann and Schmid 1984, 1985b). Since in this paper we only consider plane structures, we shall get along with two such pictures. We shall only ask for the current distributions of the  $x$  and the  $y$  components of the momentum.



**Figure 1.** (a) Electric circuit with arrows indicating the flow direction of the electric charge. (b) Conductor and current are represented separately. (c) The current has been decomposed into two partial currents.

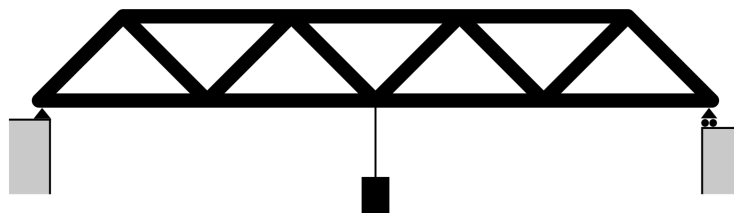
For the graphical representation of an electric current distribution, several methods may be considered, figure 1. In figure 1(a) the flow is indicated by an arrow on the side of each conductor. (The flow direction is the direction of the average of the current density vector over a cross sectional surface.) The picture of figure 1(b) is more suggestive. Here, the conductors and the currents are drawn separately. Yet another alternative is shown in figure 1(c): the total current has been decomposed in unbranched partial currents. Notice, that it is not meant that the current intensities in the various loops are the same. For the graphical representation of momentum currents we shall use the method of figure 1(c).

### 3. Calculation of the current intensities and interpretation of the stream line diagrams

In a previous paper we have shown how momentum flow distributions in static structures or trusses can be calculated (Herrmann and Schmid 1984). A truss is a system of bars which are imagined to be massless and which are joined together at their ends by means of hinges in such a way as to form a rigid structure. Sometimes, such a structure is called a ‘just-rigid’ structure in order to point out the fact that the structure is rigid, but that it loses its rigidness upon removing any single bar. How does a typical problem of the theory of trusses present itself? In order to help the reader to recognize the problem, we formulate it in the more familiar force representation instead of the momentum current representation.

A just-rigid structure is presented, as well as one force or several forces acting on the junctions of the structure. An example is the bridge-like structure in figure 2. The problem consists of finding the internal forces, i.e. the forces acting in each of the bars. As a first step in solving the problem one calculates the forces which are exerted on the structure by the supports. Thereafter, one works one’s way through the structure by means of the vector addition rule, taking profit of the fact that, thanks to the hinges, each force vector is parallel to the corresponding bar. Often one discovers that a bar is uncharged, i.e. the force is zero.

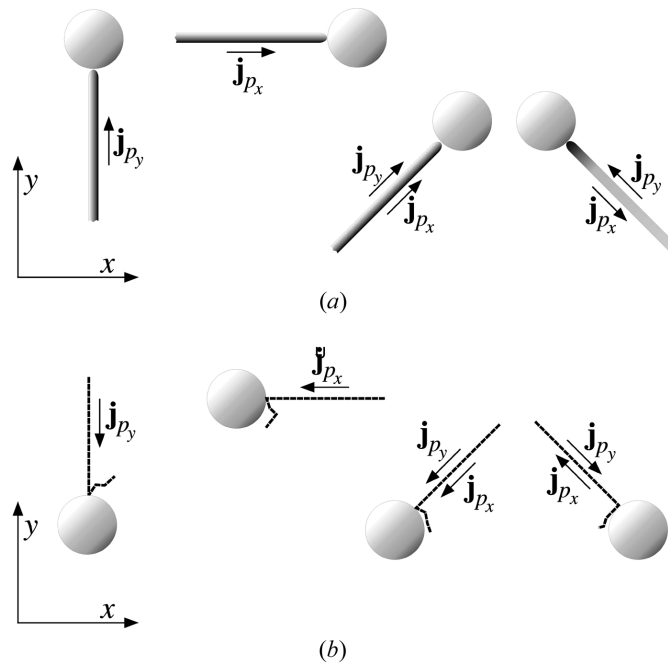
The components  $F_{i,x}$  and  $F_{i,y}$  of the force in every bar  $i$ , which are obtained in this way, are identical with the current intensities of the  $x$  and the  $y$  momenta, respectively, flowing through bar  $i$ .



**Figure 2.** Bridge with a load. Which are the forces acting in each of the bars of the truss?

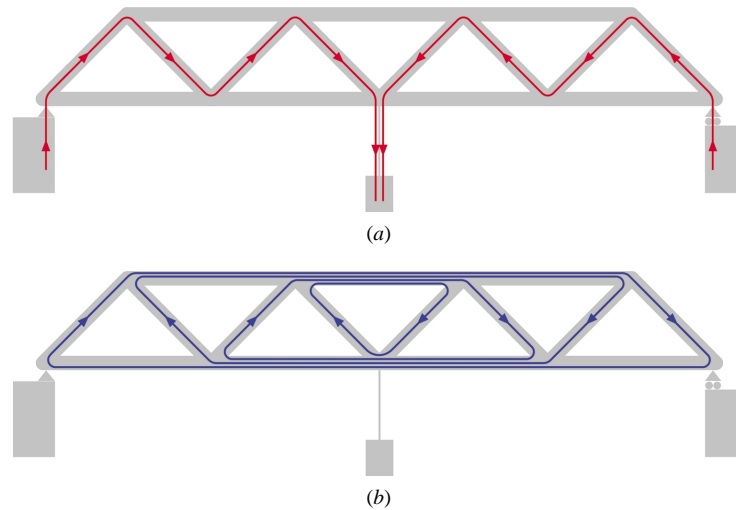
To be able to draw a flow diagram, one still needs the flow direction of the currents. In order to obtain these directions the rules depicted in figure 3 have to be applied. A body is charged with momentum by pushing, with the help of a bar, figure 3(a), or by pulling by means of a string, figure 3(b). From the resulting change of the momentum of the body one can deduce the direction of the momentum flow density vectors  $\mathbf{j}_{p_x}$  and  $\mathbf{j}_{p_y}$  in the corresponding conductor (the bar or the string). The following rule is found: if the conductor is under compressive stress,  $x$  momentum flows in the positive  $x$  direction, and  $y$  momentum in the positive  $y$  direction. If the conductor is under tensile stress,  $x$  momentum flows in the negative  $x$  direction and  $y$  momentum in the negative  $y$  direction. Consequently, in a bar that is oriented in the  $x$  direction there is no flow of  $y$  momentum and in a bar pointing in the  $y$  direction there is no flow of  $x$  momentum. A corollary of the rules is the following: in a bar which is at an angle of between  $0^\circ$  and  $90^\circ$  with the  $x$ -axis, the  $x$  and the  $y$  momenta flow in the same direction. In a bar which is at an angle of between  $90^\circ$  and  $180^\circ$  with the  $x$ -axis, the  $x$  and the  $y$  momenta flow in opposite directions. If this method is applied to the bridge of figure 2, the  $x$  and  $y$  momenta stream lines as depicted in figure 4 are obtained.

Our flow diagrams represent the flow of the components of a vector quantity. However, the components of a vector depend on the orientation of the coordinate system. Therefore, if this orientation is changed, the flow lines will also change their shape. The flow diagram will be most simple when the coordinate system is chosen in such a way that the symmetry of the structure and of the external forces is taken into account. As an example, let us consider the truss of figure 5. In figure 5(a) the coordinate system has been chosen in such a way that the



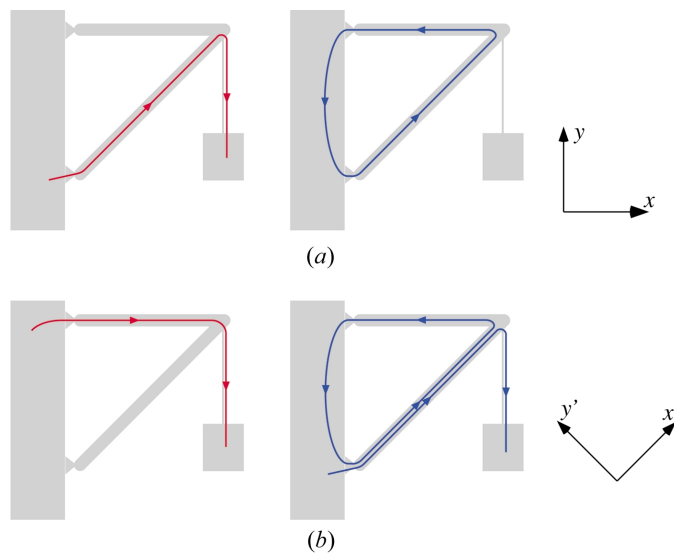
**Figure 3.** Determination of the flow direction of momentum. (a) Bars under compressive stress. (b) Strings under tensional stress.

$y$ -axis is parallel to the gravitational field. In figure 5(b) the  $y'$ -axis is at an angle of  $45^\circ$  with the gravitational field. Now, what can be seen on a momentum flow diagram? Let us once more consider the bridge of figure 4.



**Figure 4.** Momentum flow diagram of the bridge of figure 2. (a)  $y$  momentum. (b)  $x$  momentum.

- (1) We obtain information on the charge of the various bars. Figure 4(a) tells us that all of the diagonal bars are subject to the same charge. From figure 4(b) we can read that the



**Figure 5.** Momentum flow in the same structure represented with two different orientations of the coordinate system. Left-hand side:  $y$  momentum; and right-hand side:  $x$  momentum.

charge of the horizontal bars decreases when going from the centre to the right-hand and to the left-hand sides.

- (2) We can read from a flow diagram if there is compressive or tensional stress in a bar. In the diagonal bars of figure 4, there is alternately compressive and tensional stress. In the upper horizontal bars there is pressure, and in the lower horizontal bars there is tension. Obviously, one can read from a diagram when in any bar there is no stress at all.
- (3) We can see the global structure of the flow. In the case of figure 4, it is seen that the  $y$  momentum flow enters through the supports at the right-hand and at the left-hand sides, and it leaves the structure via the charge which is hanging in the middle. The  $x$  momentum, on the other hand, flows in closed loops within the structure.

#### 4. Rules for the qualitative drawing of momentum flow diagrams

Let us now compile a set of rules which will enable us to draw a momentum flow diagram qualitatively. We shall restrict ourselves to a small number of rules. It would be easy to extend and refine this canon. However, if we have too many rules we shall lose the main advantage of the method: the simplicity of dealing with them.

Since momentum is a conserved quantity and since the momentum density does not change in any element of a truss (indeed, it is always and everywhere zero), we can formulate the following.

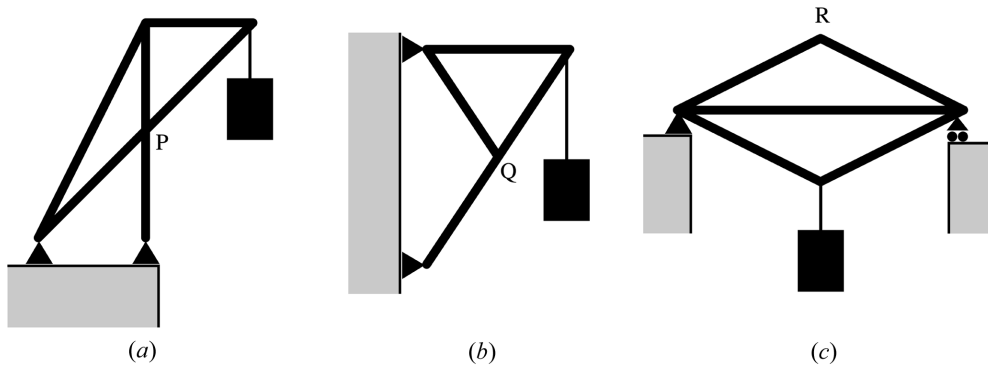
**Rule 1.** *Flow lines do not begin or end anywhere.*

A corollary of this rule is: *in a junction, for each vector component of the momentum current the junction rule holds.*

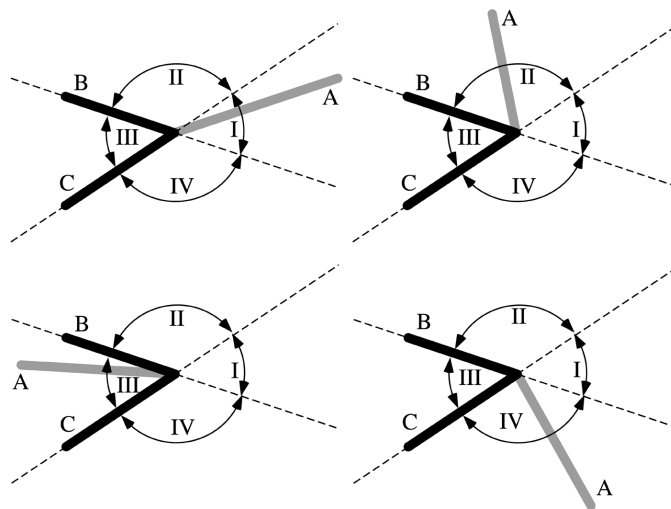
Often, one has to deal with a special case of a junction: when three bars A, B and C meet in a junction and the flow in one of them, say in A, is zero, then the flow entering via B is equal to the flow leaving the junction via C. An example is the  $x$  flow in the rightmost junction of figure 4, another one is the  $y$  flow through the same junction.

In section 3 we have already formulated rules on the relation between the direction of a flow and the tensional state of the momentum conductor.

**Rule 2.** *Pressure:  $x$  momentum flows in the positive  $x$  direction;  $y$  momentum flows in the positive  $y$  direction. Tension:  $x$  momentum flows in the negative  $x$  direction;  $y$  momentum flows in the negative  $y$  direction.*



**Figure 6.** (a) Four conductors, having the same direction two by two, meet in junction P. The current intensities in opposite conductors are equal. (b) Three conductors meet in junction Q. Two of them have the same direction. The current in the third one is zero. (c) Two bars of different directions meet in junction R. Both of them are without current.



**Figure 7.** The stress state (pressure or tension) of the bars B and C depends on: the stress state of bar A; and the quadrant I, II, III or IV into which A is oriented.

A corollary of this rule is: *in a bar that is oriented in the  $x$  direction, there is no flow of  $y$  momentum and in a bar which is pointing in the  $y$  direction there is no flow of  $x$  momentum.*

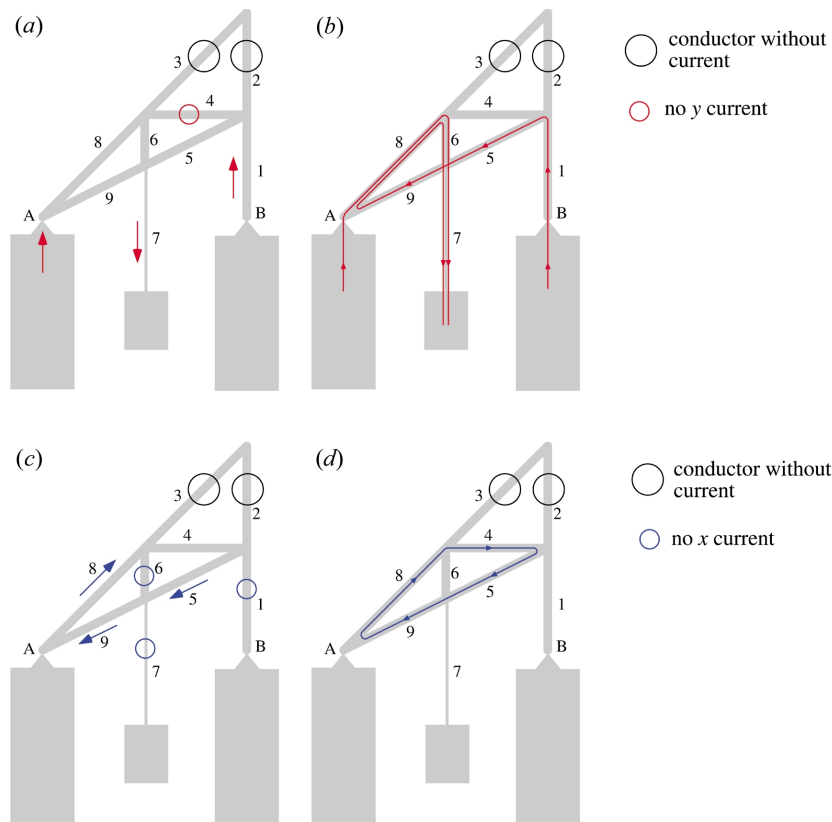
Again, figure 4 shows an example. In particular, in all the horizontal bars the  $y$  momentum flow is zero, and in the suspension string of the charge the  $x$  momentum flow is zero. A roller support is equivalent to a bar which is perpendicular to the roller surface. Therefore, rule 2 reads in this case: *a horizontal roller support blocks the flow of  $x$  momentum, a vertical roller support blocks the flow of  $y$  momentum.*

Symmetries can be helpful in several ways. Sometimes the whole structure is symmetrical, sometimes only a detail. An example of a mirror-symmetric truss is again the bridge of figure 4.

We have chosen the  $y$ -axis to be parallel to the axis of symmetry. Therefore, the  $y$  currents display the same mirror symmetry as the truss itself. For the  $x$  currents, in contrast, other symmetry results. When mirroring the  $x$  stream lines we have also to invert the direction of flow for every line. We thus have the following rule.

**Table 1.** The stress state of B and C follows from that of A and the quadrant into which A is pointing.

A in quadrant	Stress state of A	Stress state of B	Stress state of C
I	Pressure Tension	Pressure Tension	Pressure Tension
II	Pressure Tension	Tension Pressure	Pressure Tension
III	Pressure Tension	Tension Pressure	Tension Pressure
IV	Pressure Tension	Pressure Tension	Tension Pressure



**Figure 8.** Consider the figures in the order (a), (b), (c), (d). The momentum flow lines are developed by means of our rules.

**Rule 3.** *If the truss, together with the in- and outflowing momentum currents, displays a mirror axis in the  $y$  direction, then the  $y$  momentum flow diagram has the same symmetry. The  $x$*

*momentum flow lines transform into themselves when, in addition to mirroring them, the flow direction is inverted.*

An example of a local symmetry is what we shall call a ‘cross junction’: a junction of four momentum conductors with two of them pointing in the same direction and the other two in another common direction. For such a junction the following rule is applicable.

**Table 2.** The steps for drawing the flow diagrams of figure 8.

Figure	Operation	Rule
8(a)	Direction of the $y$ currents in conductors 1 and 7;	2
	direction of the $y$ current in support A;	2
	no current in conductors 2 and 3;	4
	no $y$ current in conductor 4.	2
8(b)	The $y$ stream line turns from conductor 1 into conductor 5;	1
	it goes unchanged into conductor 9;	4
	it meets the current entering via support A. Both continue into conductor 8 and bend into conductor 6;	1
	they enter unchanged the suspension 7.	4
8(c)	No $x$ current in conductors 1, 6 and 7;	2
	direction of the $x$ current in conductors 5, 8 and 9.	2
8(d)	The $x$ stream line of conductor 4 bends into conductor 5;	1
	it goes unchanged into conductor 9;	4
	Since no $x$ momentum enters the structure via support B and the load, there is no $x$ momentum flow through support A. Therefore, the $x$ flow line from conductor 9 bends into conductor 8;	
	the $x$ momentum flow line of conductor 8 bends into conductor 4.	1

**Rule 4.** *If four conductors meet in a cross junction, i.e. if they have the same direction two by two, the current intensities in opposite conductors are equal.*

The reason is easy to see. The vector sum of the four momentum flow vectors (=force vectors) is zero. If the addition is realized graphically by attaching the tip of one arrow to the end of the next one, a parallelogram is obtained. Since, in a parallelogram, opposite sides have the same direction and the same length, the absolute values of the components of the corresponding vectors are equal. An example is junction P in figure 6(a).

A special case of this rule is seen to work in the truss of figure 6(b). We can imagine completing junction Q by a fourth conductor such that a cross junction will result. Since in the fourth bar the current is zero, our rule about cross junctions tells us that also the current in the opposite bar must be zero. We thus obtain: *if in a three-conductor junction, two conductors have the same direction, then the current in the third conductor is zero.*

A second special case of the cross junction rule can be seen to operate in the structure of figure 6(c). We can complete the two-conductor junction R by two more currentless conductors such that a cross junction results. We thus obtain: *if only two conductors of different orientation meet in a junction, no momentum current flows in any of them.*

Consider now a junction of three conductors A, B and C. Since the direction of all of the bars or strings is known, we also know the direction of the momentum current vectors (force vectors) in the three conductors. Thanks to the vector addition rule we can calculate the momentum currents in, say, B and C if that of A is given. From this rule, which is needed for a quantitative analysis of the system, a ‘weaker’ rule follows which is easy to handle and which is useful for the drawing of momentum flow diagrams. This rule tells us that we can deduce the stress state of two of the bars, say B and C, from the stress state of the third conductor A. By knowing the stress state, we mean that we know if a conductor is under compressive or under tensional stress. In other words: the stress state is a one-bit variable. We explain the rule by means of figure 7. The directions of B and C divide the plane into four quadrants I, II,

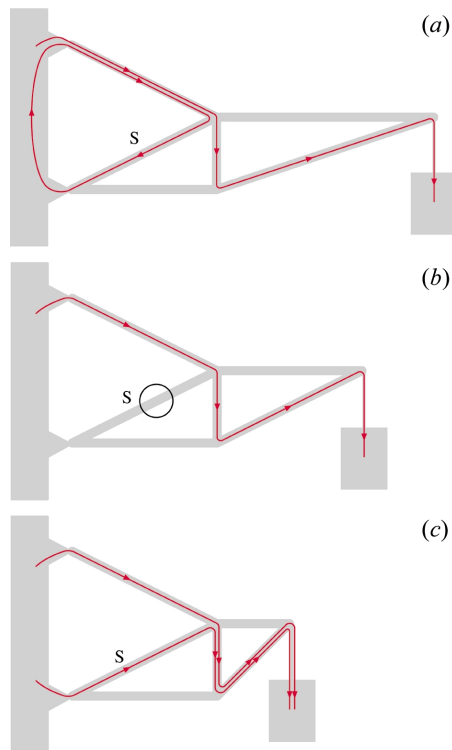


III and IV. The stress state of B and C depends upon the quadrant into which conductor A is pointing, see table 1.

Rule 5. See table 1.

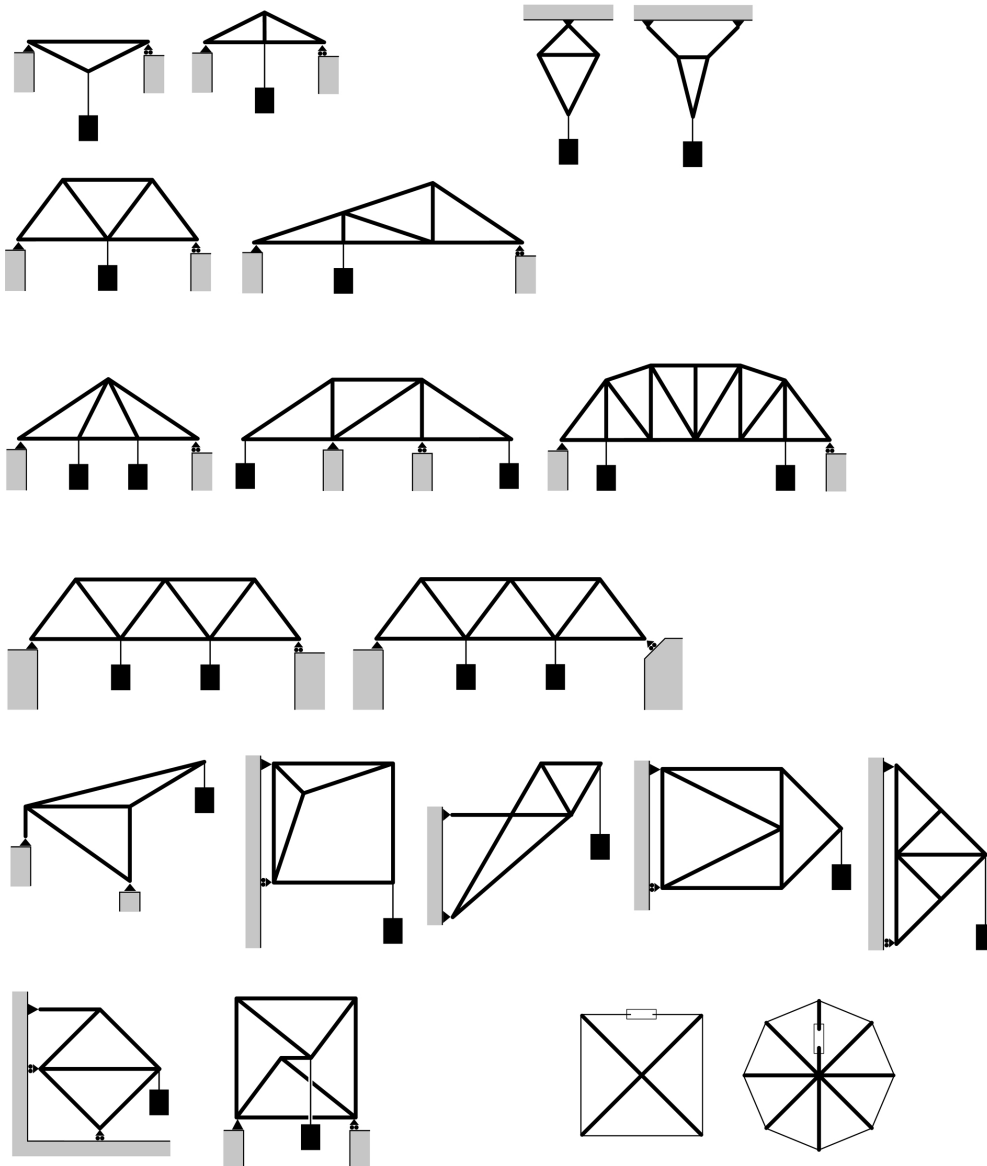
### 5. Complete solution of a problem by means of the rules

In spite of the small number of rules, we are now able to draw a complete momentum flow diagram for a great number of systems. Indeed, all of the problems of section 6 have been solved by exclusively using our set of rules. Since applying the rules to a given problem requires some skill, we shall show now, by means of an example, how a flow diagram is built step by step, figure 8. The various steps are listed in table 2. The sequence of partial pictures of figure 8 also reflects the genesis of the flow diagram.



**Figure 9.** The right section of the derrick has a different length in (a), (b) and (c). As a consequence, the direction of the momentum flow in conductor S is different. Shown is the flow of the  $y$  momentum. Notice that the current intensity corresponding to one stream line is not the same in the three figures.

Finally, we would like to show that not every problem can be solved by means of the preceding rules. Sometimes the direction of the flow depends on the exact value of a certain angle, on the exact length of a certain conductor or on the ratio of the masses of two charges. Figure 9 shows such an example. In figure 9(b) the right section of the derrick is shorter than in figure 9(a), and in figure 9(c) it is shorter than in figure 9(b). In the first case, in conductor S we have a tensional stress, in the second there is no stress at all and in the third we have compressive stress. Thus, in figure 9(a), the momentum current in S flows towards the lower, left end; in figure 9(c) towards the upper, right end and in figure 9(b) there is no flow at all.



**Figure 10.** The online version displays coloured flow diagrams for the above structures. Clicking on any of the trusses brings you to the corresponding flow diagram.

## 6. More examples

The online version contains momentum flow diagrams of the structures shown on a reduced scale in figure 10. Most of the examples stem from text books (Timoshenko and Young 1956, Langhaar and Boresi 1959). We have chosen them in such a way that every example displays some peculiarity. Among them are simple and complicated systems, symmetrical and unsymmetrical structures, trusses with a single load and trusses with several loads. There are horizontal, vertical and inclined supports. The momentum flows are caused by a hanging mass or by a turnbuckle. In the online version, ‘clicking’ on any one of the trusses represented in figure 10 brings you to the corresponding flow diagram.

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