Understanding the stability of stars by means of thought experiments with a model star

F. Herrmann and H. Hauptmann
Abteilung für Didaktik der Physik, Universität, 76128 Karlsruhe, Germany

The stability of the nuclear fusion reaction in a star is due to the negative specific heat of the system. Examining the literature one gets the impression that this phenomenon results from a complicated interplay of the various pertinent field variables. We introduce a simple model system, which displays the same behavior as a star and which can be treated quantitatively without solving any differential equation. © *1997 American Association of Physics Teachers.*

I. INTRODUCTION

The development of life on the earth was possible because the sun has been shining steadily and regularly for billions of years. From the viewpoint of physics, this behavior of the sun is rather strange. All nuclear fuel, i.e., hydrogen, is stockpiled within the sun. When comparing the reaction in the sun with a terrestrial combustion, it would correspond to a stove that has been fed with the fuel and oxygen reserve for its whole lifetime—indeed an explosive mix. One equally might compare the sun with the reactor of a nuclear power plant. As a matter of fact, the nuclear fuel for several years is charged into the reactor all at once. However, the reactor is equipped with control rods which are part of an active feedback system that guarantees a constant reaction rate.

But who takes care that the reactions in the sun or in any other sunlike star are running so steadily? What feedback mechanisms prevent the sun from exploding like a gigantic hydrogen bomb? It is due to what sometimes is referred to as a "negative specific heat."[1-4] This term means that a system's temperature decreases when heat, i.e., energy and entropy, is supplied to it. At the same time, the volume of the system increases. Now, if the energy production rate makes an excursion toward higher values, the temperature will decrease and thus the reaction will slow down. If the energy production rate deviates toward lower values, the temperature will increase and the reaction rate will again be corrected.

When consulting a typical textbook about astrophysics, one gets the impression that it is not easy to understand this behavior. To describe the star, several variables are chosen: temperature, pressure, density, energy production rate, luminosity (=energy flow), opacity and mass. The following laws are needed, which relate these variables to one another: the perfect gas law, the law of gravitation, conservation laws of mass and energy, the condition for hydrostatic equilibrium and the law of Stefan and Boltzmann.[1-6] Moreover, some approximations are needed and some adaptable parameters are introduced. All this is put into the mathematical mill. The result is a comprehensive description of the mechanics and thermodynamics of the star.

According to these derivations, the stability of a star seems to be the result of a complicated interplay of many variables, and related to the particular distribution of the values of these variables as a function of r, the distance from the center of the star. Such a calculation, which takes into account the actual constitutive relations, is indispensable when numerical results, or at least orders of magnitude, are needed. But when trying to understand the underlying physics, this may be inappropriate. In order to understand a phenomenon it is best to consider the simplest conditions under which it can occur. In this way one doesn't only learn what a phenomenon depends on, but also—just as important—what it does not depend on.

The complexity of the textbook derivations has frequently been deplored. Celnikier writes in the introduction to his article:[7] "...analytical analyses are often obscure with little obvious relevance to real stars, while numerical models of
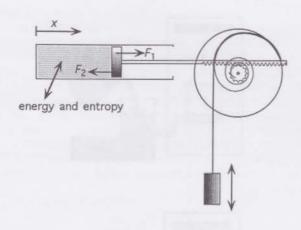
Fig. 1. Model of a star. The equilibrium position of the piston depends on the energy content of the gas. When supplying energy (and entropy) to the gas the piston moves to the right, i.e., the volume increases, and the temperature of the gas decreases.

realistic objects are so complicated and so full of parameters that the physical basis on which they are built often disappears from sight.''

In Refs. 7–11 several valuable alternatives are suggested. But since the authors are primarily interested in stellar structure, the question of stellar stability appears to be linked to several other problems. The derivations proposed in Refs. 7–11 are still too complicated for somebody who wants no more than a simple answer to the simple question of why a star is so stable.

That is why we asked the following questions: Isn't it possible to understand the negative specific heat of the sun without considering field variables? Isn't it possible to observe the same behavior in a homogeneous system? Isn't it possible to understand the behavior of a star without solving a system of four differential equations? The answer we found is: Yes, it is.

In Sec. II a simple laboratory model of the sun will be introduced. This model system can be treated quantitatively with only some simple algebra. Just as the sun, the model system's temperature decreases when energy and entropy are supplied to it. In Sec. III the energy and entropy balances of our model star are discussed.

II. OUR LABORATORY SUN

A. Description of the model

Figure 1 shows our model star. Naturally, we did not try to build it in reality. It would be difficult to get it to work because of the friction of the piston and because of heat losses of the gas in the cylinder. We will assume the gas to be ideal and will invoke standard ideal-gas results.

How does the model work? Whereas in a real star the gas is held together by the gravitational field, the gas in our model star is held together by a cylindrical container with a piston.

We assume that we can control the heat flow, i.e., the flow of energy and entropy, into or out of the gas. When the gas is heated, the energy flow P entering the gas is related to the entropy flow I_s entering it by[12,13]

$$P = TI_s.$$

As long as energy is flowing into the gas in the form of heat, another energy current is leaving it via the piston. The incoming entropy, on the contrary, remains stockpiled within the gas.

Two forces act on the piston: the force $F_1(x)$ of the gas and the force $F_2(x)$ of the weight-and-pulley arrangement on the right of Fig. 1, where x is the length of the column of gas.

By appropriately shaping the groove on which the string is winding or unwinding, a particular force law $F_2(x)$ can be realized.[14] We choose the force law to be

$$F_2(x) = -\frac{C_2}{x^{\alpha}},$$

where C_2 is a positive constant and

$$1 < \alpha < \gamma. \tag{1}$$

Here, γ is the adiabatic exponent of the gas.

B. The mechanical equilibrium of the system

We begin by discussing the mechanical stability of the system. Therefore, for the moment, we prevent any heat exchange of the gas. Thus, the entropy of the gas is held constant and the $p-V$ relationship for the gas is

$$pV^{\gamma} = \text{const}.$$

With $F_1 \propto p$ and $V \propto x$ we get the force law of the gas

$$F_1(x) = \frac{C_1}{x^{\gamma}},$$

where C_1 is a positive constant depending on the entropy content of the gas.

The condition for mechanical equilibrium of the piston

$$F(x_0) = F_1(x_0) + F_2(x_0) = \frac{C_1}{x_0^{\gamma}} - \frac{C_2}{x_0^{\alpha}} = 0 \tag{2}$$

can be fulfilled, whatever the values of C_1, C_2, α and γ.

To show that this is a stable equilibrium, we calculate the derivative of $F(x)$ at $x = x_0$,

$$\left.\frac{dF(x)}{dx}\right|_{x_0} = -\gamma \frac{C_1}{x_0^{\gamma-1}} + \alpha \frac{C_2}{x_0^{\alpha-1}} = \frac{C_1}{x_0^{\gamma-1}}(-\gamma + \alpha).$$

In the second step, Eq. (2) has been used.

Since $\alpha < \gamma$, it follows that

$$\left.\frac{dF(x)}{dx}\right|_{x_0} < 0,$$

i.e., for a small deviation from the equilibrium position the system will be driven back to x_0.

C. The behavior of the system under heat exchange

When supplying heat to, or extracting it from our gas, the piston will move but always remain in states of mechanical equilibrium. Let us calculate the $p-V$ relationship of the gas for these states. With

$$F_1(x) = -F_2(x),$$

we get

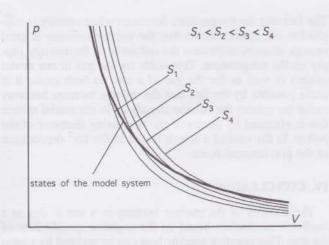$$F_1(x) = \frac{C_2}{x^{\alpha}}.$$

Fig. 2. The bold curve represents the stable states of the model system. When following this curve from smaller to greater volumes, the curve intersects adiabats (=isentrops) belonging to increasing entropy.

and with $F_1 \propto p$ and $V \propto x$, it follows that

$$p(V) = \frac{C}{V^\alpha},$$

where C is a constant. The exponent α lies between that of an adiabat and an isotherm. Such a relationship is called a polytrope. Let us remember that it represents states of mechanical equilibrium that are distinguished by different entropy contents of the gas. The system is running through these states when heat is supplied to or extracted from the gas. These states are shown as bold lines in the $p-V$ diagrams of Figs. 2 and 3.

Figure 2 shows these states together with a series of adiabats. It is seen that, when following the $p-V$ curve of our model sun from smaller to greater volumes, the lines of constant entropy, which it intersects, belong to increasing entropy. Thus, when supplying entropy to the gas, the volume increases. Since the entropy supply is related to a heat supply, we can conclude: *When heat is supplied to the gas its volume increases.*

Figure 3 shows the $p-V$ curve of our gas together with a series of isotherms. When moving on this curve toward
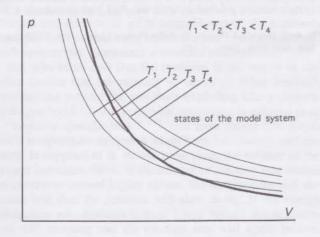

Fig. 3. The bold curve represents the stable states of the model system. When following this curve from smaller to greater volumes, the curve intersects isotherms belonging to decreasing temperatures.

higher volumes, we intersect isotherms of decreasing temperature. In other words: *When heat is supplied to the gas its temperature decreases.*

Thus our model system qualitatively behaves like a star. Notice that in the experiments we are doing with our model system, all of the processes are reversible. In a real star there are highly irreversible processes going on: in particular, the fusion reaction itself and the heat transfer from the reaction zone outwards. As our modeling shows, however, these irreversibilities are not related to the stability of the star.

In the following section we will discuss the energy and the entropy balance of the model and compare it with that of a real star.

III. THE ENERGY AND THE ENTROPY BALANCE

A. The energy balance

Our system consists of two interacting subsystems, both of which can store energy: (1) the gas. Its energy is called internal energy; and (2) the weight-and-pulley arrangement on the right.

Now, when energy is supplied to the gas in the form of heat, we cannot conclude that this energy remains in the gas, since the subsystem "gas" is interacting with the subsystem "weight-and-pulley." Indeed, we have seen that when supplying heat to the gas, its temperature decreases. Since for an ideal gas the internal energy depends only on the temperature, supplying heat to the gas causes its internal energy to decrease. In other words, we supply energy to the gas, but its energy content decreases. Although it sounds strange, this is not paradoxical, since the gas is connected to our second subsystem. We thus conclude that when supplying a certain amount of energy to the gas, more than this amount is passed over to the weight-and-pulley subsystem.

A similar process is going on in a star. A star also can be decomposed into two interacting subsystems. One is the star's matter, or the "gas," the other is its gravitational "field." Now imagine, for a while, that we are able to control the energy input of the star from the fusion reaction and the energy output via the radiation. Let us suppress the energy output and consider what happens with the energy supplied by the nuclear fusion reaction.

The energy of fusion is first supplied to the gas, and one might expect that the energy of the gas will increase. However, this is not what actually happens. When supplying a certain amount of energy to the gas, more than this amount is passed over to the other subsystem, the gravitational field, leading to an expansion of the star. In fact, the amount forwarded to the gravitational field is just twice that supplied to the gas in the first place, as follows from the virial theorem.

We are accustomed to observing that the amount of any conserved quantity increases when we add a certain amount of it. In order to show our students that this is not necessarily so, we perform an experiment which is as simple as it is nice (see Fig. 4). Two containers of about 1 liter are connected via a flexible tube.[15] One of them is suspended by a soft spring. The water level is the same in both containers. We hide from the eyes of the students the right container and the spring by means of a piece of cardboard. The demonstration consists in pouring water into the container which is visible to the students. Surprisingly, the water level will go down instead of up. We then take the cardboard away and discuss how the device works.
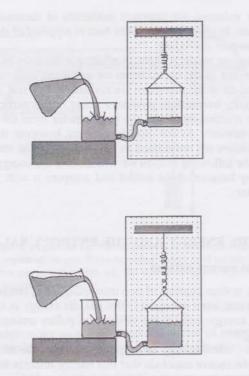
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Fig. 4. The container on the right is hidden from the students. When pouring water into the container at the left, the water level will decrease.

B. The entropy balance

The entropy balance is simpler since we have to consider only one of the two subsystems. Neither the weight-and-pulley subsystem in our model nor the gravitational field in the real star can store entropy.

When discussing the entropy balance of a gas, we have to take into account that the entropy depends on two variables: the volume and the temperature of the gas. The greater the volume and the higher the temperature of a gas, the more entropy it contains. (Both dependencies are logarithmic.[16] From statistical physics it is easily understood that the entropy increases with both the volume and the temperature: An increase of the volume as well as an increase of the temperature cause the occupied phase space to become larger.) Thus, when asking for the entropy of a gas one has to consider both its volume and its temperature. In the preceding section we have seen, that, when supplying entropy to the gas; its volume increases; and its temperature decreases.

Under "normal" conditions, i.e., when the volume is kept constant, an entropy supply causes the temperature to rise.

The fact that the temperature decreases when entropy is supplied to the system shows that the volume increase is great enough to overcompensate the influence of the entropy supply on the temperature. This holds for the gas in our model system as well as for the gas of a star. In both cases, it is made possible by the fact that the volume increase becomes easier the greater the volume already is. In our model system this is obtained by means of the decreasing diameter of the pulley. In the case of a star, it is due to the $1/r^2$ dependence of the gravitational force.

IV. CONCLUSION

The stability of the nuclear burning in a star is due to a feedback mechanism based on the negative specific heat of the star. This negative specific heat can be realized by means of a simple model system. With such a model, the mechanics and thermodynamics of a star can be understood qualitatively without any recourse to field differential equations.

[1]R. Kippenhahn and A. Weigert, *Stellar Structure and Evolution* (Springer-Verlag, Berlin, 1990), pp. 17 and 243.

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[13]H. U. Fuchs, *The Dynamics of Heat* (Springer-Verlag, New York, 1996), p. 88.

[14]It may not seem evident that a force law satisfying Eq. (1) can be realized with this device. Indeed, the proof is a bit lengthy, but it can be obtained from the authors. Of course, one could also imagine another mechanical device which does the same job.

[15]In order to get a descending water level the following inequality must hold: $\frac{1}{2}A\rho g < D < A\rho g$ where D is the spring constant, A the cross section of the containers, ρ the density of the water and g the gravitational acceleration.

[16]F. Reif, *Statistical Physics*, Berkeley Physics Course, Vol. 5 (McGraw-Hill, New York, 1965), p. 278.