The unexpected path of the energy in a moving capacitor

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(Received 9 September 1991; accepted 25 June 1992)

Most physics students know little about energy flow distributions. The example shows the unexpected path of the energy in a familiar situation.

Every physics student acquires competence in predicting the distribution of various vector fields: He or she is able to draw electric field lines of some simple but typical charge distributions; he or she becomes acquainted with some typical magnetic field distributions. The student also gets a good qualitative understanding of the distribution of current densities of electric currents and of mass currents in a flowing liquid. However, the student learns much less about energy flow fields, i.e., the flow of that quantity which is considered to be one of the fundamental quantities of modern physics. In the past, the *American Journal of Physics* has contributed a lot to alleviating this shortcoming.[1–7]

In the present article, we will explore the energy flow in a charged moving capacitor. Since the electric and the magnetic fields between the plates of the capacitor are homogeneous, the problem is mathematically very simple. It is, however, conceptually interesting since it displays a surprising peculiarity. We will limit ourselves to the discussion of two special cases: (i) the movement in a direction perpendicular to the plates and (ii) the movement parallel to the plates.

Let us call the distance between the plates d and the two lateral dimensions of the plates s and l, respectively, see Fig. 1. The space between the plates is filled with electric field of strength $\mathbf{E}$. There are two spacers to keep the plates apart. The energy contained within the field between the plates is

$$W=(\epsilon_0/2)\,|\mathbf{E}|^2 lsd.$$

A. Movement perpendicular to the plates

Let the capacitor of Fig. 1 move with velocity $\mathbf{v}_\perp$ perpendicularly to its plates, i.e., in the positive z direction. Consider a reference surface A_z that is parallel to the plates and halfway between them, and calculate the rate of change dW/dt of the energy on the upper side of this surface:

$$\frac{dW}{dt}=\frac{\epsilon_0}{2}\,|\mathbf{E}|^2|\mathbf{v}_\perp|\,ls. \quad (1)$$

The rate of change of the energy on the lower side of the reference surface is the negative of this expression. One might now suspect that the energy simply moves together with the plates, just like water moves together with the walls of its container when the container is moving. Thus dW/dt should be equal to the flow of the energy within the electromagnetic field through the surface A_z.

Whenever energy is flowing within the electromagnetic field, the energy current density can be identified with the Poynting vector $\mathbf{E}\times\mathbf{H}$. Multiplying the z component of $\mathbf{E}\times\mathbf{H}$ by the area ls of our horizontal reference surface yields the energy current flowing in the field from below to above the reference surface. This current should, according to our expectations, be equal to the rate of change of the energy above the reference surface, Eq. (1).

Let us do the calculation. First we need to know the strength of the magnetic field between the plates. The movement of the capacitor can be considered the result of a change of the reference frame. A capacitor at rest in reference frame S' is moving at velocity $\mathbf{v}$ in a reference frame S, if S is moving at velocity $-\mathbf{v}$ against S'. To get the magnetic field strength in the moving capacitor one can use the corresponding transformation law:[8]

$$\mathbf{H}=\epsilon_0\mathbf{v}\times\mathbf{E}. \quad (2)$$

Using

$$\mathbf{v}=\mathbf{v}_\perp,$$

we get

$$\mathbf{H}=\epsilon_0\mathbf{v}_\perp\times\mathbf{E}=0.$$

Since the velocity is parallel to the electric field strength vector, there is no magnetic field. Consequently, the Poynting vector is zero everywhere on our reference surface and, therefore, no energy flows across it. Thus we are left with the question: How does the energy get from the lower to the upper part of the capacitor?

Until now, we didn't take into account that there is yet another connection crossing the reference surface A_z: the spacers. The spacers are under compressional stress and they are moving upward. As a result they transmit energy from the lower to the upper part of the system just as we would transmit energy to a wagon when pushing against it from behind by means of a rod. The energy flow in the spacers is obtained by the expression

$$P=\mathbf{v}_\perp\cdot\mathbf{F}. \quad (3)$$

Here, $\mathbf{F}$ is the force exerted by the lower plate on the upper plate via the spacers. It is equal and opposite to the force $\mathbf{F}_{\perp\,\text{field}}$ which is exerted by the lower plate onto the upper one via the field, and which is due to the tensile stress in the direction of the field lines:

$$\mathbf{F}=-\mathbf{F}_{\perp\,\text{field}}.$$

The force $\mathbf{F}_{\perp\,\text{field}}$ is obtained by multiplying the negative of the zz component σ_z of the stress tensor of the field by the vectorial surface area $ls\mathbf{e}_z$ ($\mathbf{e}_z$ being the unit vector in z direction).[9]

$$\mathbf{F}_{\perp\,\text{field}}=-\sigma_z ls\mathbf{e}_z.$$

The minus sign in this expression is due to the convention that tensile stress is associated to a positive value of the diagonal components of the stress tensor.

Since the stress is

$$\sigma_z=(\epsilon_0/2)\,|\mathbf{E}|^2,$$

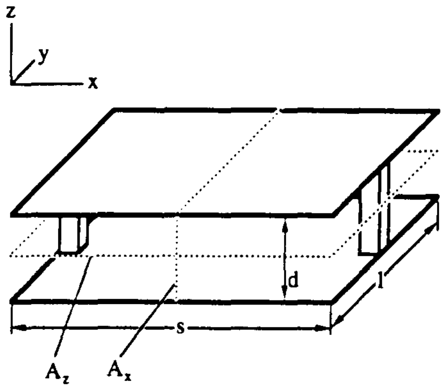

Fig. 1. When the capacitor is moving upward, energy is flowing through the spacers from the lower to the upper plate. When the capacitor is moving to the right, energy is flowing through the plates from right to left.

we get

$$\mathbf{F}=(\epsilon_0/2)|\mathbf{E}|^2 ls\mathbf{e}_z,$$

and with Eq. (3),

$$P=(\epsilon_0/2)|\mathbf{E}|^2|\mathbf{v}_\perp|ls,$$

i.e., the value required by Eq. (1). We see that the transfer of the energy from the lower to the upper part of the field between the plates of the capacitor does not take place via the field itself. The energy is "collected" by the lower plate while this plate is moving towards the region that contains the field. From there the energy flows through the material parts of the system to the upper plate. The upper plate "deposits" the energy in the region below it, whereby new field is formed.

We got this result by establishing a global energy balance: The rate of change of the energy above our reference surface must be equal to the energy flow through this surface. Thus we have taken advantage of the global energy conservation. This kind of argument has the advantage of being mathematically simple because we don't need to know the whole energy flow distribution. However, if we are interested in these details we have to apply the local continuity equation of the energy:[10]

$$\frac{\partial u}{\partial t}+\operatorname{div}\mathbf{S}=-\mathbf{j}\cdot\mathbf{E}'.$$

Here, u represents the energy density in the electromagnetic field. In the term $\mathbf{j}\cdot\mathbf{E}'$, $\mathbf{j}$ is the current density due to the movement of the upper plate, and $\mathbf{E}'$ is the electric field due to the charge of the lower plate alone. Thus, $\mathbf{E}'$ is not the field of the whole capacitor but half of it: $\mathbf{E}'=\mathbf{E}/2$.

The energy density between the plates (not including the inner surfaces of the plates) doesn't change with the time. The Poynting vector's divergence is zero and there is no electric current. Thus the local continuity equation is satisfied in a trivial manner. The only place where it tells us something interesting is at the capacitor's plates. Here, too, div S is zero. However, the time derivative of u as well as the term $\mathbf{j}\cdot\mathbf{E}'$ is nonzero. It is a straight-forward calculus to show that $\partial u/\partial t$ and $-\mathbf{j}\cdot\mathbf{E}'$ are equal. There is a point, however, that one must pay attention to: If the surface charge of the capacitor's plates is supposed to have zero thickness, $\partial u/\partial t$ as well as $-\mathbf{j}\cdot\mathbf{E}'$ become infinite. Thus the charge should be distributed on a layer of finite thickness.

B. Movement parallel to the plates

Let the capacitor in Fig. 1 now move parallel to its plates, i.e., in the positive x direction. Its velocity is $\mathbf{v}_\parallel$. We now choose a reference surface A_x perpendicular to the plates. Consider the rate of change dW/dt of the energy on the right side of the reference surface:

$$\frac{dW}{dt}=\frac{\epsilon_0}{2}|\mathbf{E}|^2|\mathbf{v}_\parallel|ld. \tag{4}$$

The rate of change of the energy on the left side of the reference surface is the negative of expression (4).

Again we compare this rate of change of the energy with the energy flow within the field—if there is any. Applying the transformation law (2) to this case, we get a nonzero magnetic field since the direction of movement is perpendicular to the direction of the electric field:

$$\mathbf{H}=\epsilon_0\mathbf{v}_\parallel\times\mathbf{E}.$$

Thus the Poynting vector at any point within the homogeneous field between the plates is

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}=\epsilon_0|\mathbf{E}|^2\mathbf{v}_\parallel,$$

and the energy current strength turns out to be

$$P'=|\mathbf{S}|ld=\epsilon_0|\mathbf{E}|^2|\mathbf{v}_\parallel|ld. \tag{5}$$

Comparing this expression with Eq. (4), we see that the energy flow in the field from left to right is twice the rate of change of the energy on the right side. Again there is disagreement. And again we ask ourselves how the energy balance can be restored. In other words: Where does the excess energy which flows through the field go?

Here too, besides the field, we have an additional connection crossing A_x: the plates of the capacitor. The plates are under tensile stress and they are moving to the right. Thus they transmit energy from right to left just like a drive belt would do, according to the equation

$$P''=\mathbf{v}_\parallel\cdot\mathbf{F}. \tag{6}$$

The force $\mathbf{F}$ in Eq. (6) equals the sum of the forces in both plates acting in the x direction. It must be equal and opposite to the force $\mathbf{F}_{\parallel\,\text{field}}$ within the field in x direction which is due to the compressional stress perpendicular to the field lines:

$$\mathbf{F}=-\mathbf{F}_{\parallel\,\text{field}}.$$

The force $\mathbf{F}_{\parallel\,\text{field}}$ is obtained by multiplying the negative of the xx component σ_x of the stress tensor of the field by the vectorial surface area $ld\mathbf{e}_x$ ($\mathbf{e}_x$ being the unit vector in x direction):[9]

$$\mathbf{F}_{\parallel\,\text{field}}=-\sigma_x ld\mathbf{e}_x.$$

Since the stress is

$$\sigma_x=-(\epsilon_0/2)|\mathbf{E}|^2,$$

we get

$$\mathbf{F}=-(\epsilon_0/2)|\mathbf{E}|^2 ld\mathbf{e}_x. \tag{7}$$

This is the force within the plates at our perpendicular cross section: the force which the part left of the cross

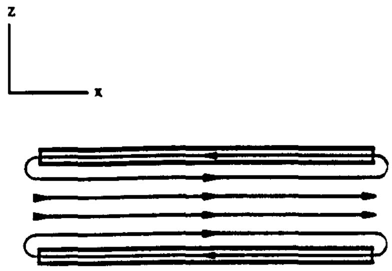

Fig. 2. Half of the energy which is moving within the field from left to right flows back to the left through the plates of the capacitor.

section exerts on the part right of it. One might ask, however, where this force originates: How is the field able to exert a force on the left side of the plates, and how can the right part of the plates exert a force on the field? The answer can only be: By means of the stray field. Indeed, only the stray field at both ends of the plates has components parallel to the plate direction and is therefore able to pull outwards.[11]

Inserting (7) in (6) we get the energy flow within the plates:

$$P'' = -(\epsilon_0/2)|\mathbf{E}|^2|\mathbf{v}_{\parallel}|ld. \tag{8}$$

Now the energy balance is restored. Indeed, by using Eqs. (4), (5), and (8) it can be seen that

$$\frac{dW}{dt} = P' + P''.$$

Half of the total energy current which goes from left to right in the field flows back through the plates and the other half serves to transfer the electrostatic field from left to right, see Fig. 2. That part of the energy current within the field which goes back through the plates has to make 180° turns within the stray field on the right side of the capacitor in order to get into the plates, and it must make additional turns when it leaves the plates on the left side of the capacitor. Here again, one recognizes the importance of the often neglected stray field.

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[11]If forces are interpreted as momentum currents and mechanical stress as momentum current density, then it can be seen that we simply have closed loops of momentum currents: Momentum is flowing from left to right within the field. It then makes 180° turns within the stray field and enters the plates. It then flows back to the left within the plates. At the left end it leaves the plates, enters the field, makes again 180° turns within the stray field region and flows to the right within the homogeneous part of the field. See F. Herrmann and G. Bruno Schmid, "Momentum flow in the electromagnetic field," Am. J. Phys. **53**, 415–420 (1985).