Gravitoelectromagnetism: Removing action-at-a-distance in teaching physics

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Gravitation is still taught largely in a way that suggests the existence of action-at-a-distance. A theory without such shortcomings, gravitoelectromagnetism, was proposed by Heaviside in 1893, but it did not become well-established because many effects it describes are very small and the later emergence of general relativity seemed to make a theory of gravitoelectromagnetism superfluous. We argue that gravitoelectromagnetism still retains relevance in the physics curriculum because it by no means describes only tiny effects and does not demand the mathematical level of general relativity.

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I. INTRODUCTION

Anyone teaching gravity who does not wish to invoke general relativity does so in essentially the way Newton formulated it. Newtonian mechanics originated at a time when there was no concept of physical fields and one had to be satisfied with invoking action-at-a-distance. Similarly, Newton did not say anything about energy for it did not yet exist as a physical quantity, and even after it was introduced, it could not be localized; that is, it was not possible to specify an energy density and an energy current density until about the end of the 19th century (Ref. 1, pp. 1180–1181). Even today, the language of energy still retains the fingerprints of its action-at-a-distance origins: If a body is lifted, we say that the potential energy of the body increases, knowing full well that this energy is not stored within the body.[2] We can circumvent action-at-a-distance by speaking of a "gravitational field," but this leaves unanswered questions as to its energy content, mechanical stress, and how the energy flows within the field. One does not typically discuss, for example, the energy density of the field or what path the energy takes from the field to a falling body.

The later development of the field concept by Faraday and Maxwell in the theory of Electromagnetism (EM) could have served as a model for a more modern theory of gravitation. Such a theory, Gravitoelectromagnetism (GEM), was in fact formulated by Heaviside and is structured in close analogy to Maxwell's theory.[3] GEM closes the gap between Newtonian mechanics and general relativity. Like EM in vacuum, GEM operates with two field strengths. One of them is the gravitational field strength **g**, which is also called the gravistatic field strength and whose source is the mass density; this is analogous to the electric field strength **E**, whose source is the electric charge density. The other is the gravinetic field strength **b**, whose source is the mass current density; this is analogous to the magnetic **B**-field, whose source is the electric current density. We have adopted the terms gravistatic and gravinetic from Krumm and Bedford.[4]

While it is common to introduce **g** when treating gravitation, the gravinetic field strength **b** is not usually mentioned because its effects are typically miniscule. As an example, we consider the rotating Earth, which generates a **b**-field and thus exerts a force on, say, an airplane. If we assume that an aircraft of 400 tonnes (400 000 kg) is flying at 1000 km/h due east at the equator, this results in a "gravitational Lorentz force" of about 1 μN; that is, the aircraft effectively becomes about 0.1 mg lighter. This calculation is detailed in Appendix A.

In our opinion, the smallness of GEM effects should not be an argument against introducing students to **b**. The action of the gravitational Lorentz force is an interesting phenomenon and gives us a better understanding of gravitation by revealing an analogy. A more important reason to introduce **b** is that it is a factor in the energy current density of the gravitational field, and here, its effect is not at all tiny. The reason for this is that while **b** enters quadratically into the forces (i.e., the momentum currents), it enters linearly into the energy current. Thus, we believe that GEM still has merit today and that it deserves a place in the pedagogical curriculum. Several other authors have advocated this as well; see, for example, Refs. 4 (p. 362), 5 (p. 889), and 6 (p. 422).

In Sec. II, we compare the equations of GEM with those of EM, and in Sec. III, we explore some of the consequences of these differences via examples, focusing on mechanical stresses (momentum current density) and the energy density and energy current density of the gravitational field. Section IV offers a few concluding remarks.

It is important to point out that three years after the appearance of general relativity, Thirring showed that a gravitoelectromagnetic description of gravitational effects emerges by linear approximation of Einstein's equation and for small velocities.[7] A comprehensive overview of the historical development of this description can be found in Iorio and Corda[8] (pp. 2–5) and is discussed in numerous publications.[8–11] While the Heaviside approach is a theory in flat space and therefore cannot describe general relativistic effects like gravitational waves or the Lense–Thirring effect, the Thirring approximation constructs GEM fields in such a way that such effects can be described locally. One recognizes that these are two different theories by the fact that according to Thirring's theory, a factor of 4 occurs in the "Lorentz force" equation, which also propagates into the energy current density (Poynting vector). In contrast, this factor does not exist in the Heaviside theory, which we explore.

As this paper was being finalized for publication, we became aware of another paper to be published in this journal which examines energy flows in gravitational fields from a Newtonian perspective where fields respond instantly to the motions of charges and masses, an approach that contrasts with our formulation.[12] Curiously, *both* approaches are able to explain energy flows. We encourage readers to examine both papers for a full picture of this historically important issue.

II. THE EQUATIONS

In this section, we compile the most important equations of EM and GEM, beginning with the four Maxwell equations and their GEM analogs. These are followed by expressions for the energy density, the energy current density, the force (momentum current), and the mechanical stress (momentum current density). In the case of EM, the force is the Lorentz force and the energy current density is the Poynting vector.

Maxwell's equations (EM, GEM):

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho_Q, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}_Q + \frac{1}{c^2}\cdot\frac{\partial \mathbf{E}}{\partial t}, \tag{4}$$

$$\nabla \cdot \mathbf{g} = -\frac{1}{\varepsilon_g}\rho_m, \tag{5}$$

$$\nabla \cdot \mathbf{b} = 0, \tag{6}$$

$$\nabla \times \mathbf{g} = -\frac{\partial \mathbf{b}}{\partial t}, \tag{7}$$

$$\nabla \times \mathbf{b} = -\mu_g \mathbf{j}_m + \frac{1}{c^2}\cdot\frac{\partial \mathbf{g}}{\partial t}. \tag{8}$$

Energy densities (ED, GEM):

$$\rho_E = \frac{\varepsilon_0}{2}\mathbf{E}^2, \tag{9}$$

$$\rho_E = \frac{1}{2\mu_0}\mathbf{B}^2, \tag{10}$$

$$\rho_E = -\frac{\varepsilon_g}{2}\mathbf{g}^2, \tag{11}$$

$$\rho_E = -\frac{1}{2\mu_g}\mathbf{b}^2. \tag{12}$$

Energy current densities (ED, GEM):

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}, \tag{13}$$

$$\mathbf{S} = -\frac{1}{\mu_g}\mathbf{g}\times\mathbf{b}. \tag{14}$$

Forces (momentum currents) (ED, GEM):

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \tag{15}$$

$$\mathbf{F} = m(\mathbf{g} + \mathbf{v}\times\mathbf{b}). \tag{16}$$

Mechanical stresses in the direction of the field lines (ED, GEM):

$$\sigma_\parallel = \frac{\varepsilon_0}{2}\mathbf{E}^2, \tag{17}$$

$$\sigma_\parallel = \frac{1}{2\mu_0}\mathbf{B}^2, \tag{18}$$

$$\sigma_\parallel = -\frac{\varepsilon_g}{2}\mathbf{g}^2, \tag{19}$$

$$\sigma_\parallel = -\frac{1}{2\mu_g}\mathbf{b}^2. \tag{20}$$

Mechanical stresses perpendicular to the field lines (ED, GEM):

$$\sigma_\perp = -\frac{\varepsilon_0}{2}\mathbf{E}^2, \tag{21}$$

$$\sigma_\perp = -\frac{1}{2\mu_0}\mathbf{B}^2, \tag{22}$$

$$\sigma_\perp = \frac{\varepsilon_g}{2}\mathbf{g}^2, \tag{23}$$

$$\sigma_\perp = \frac{1}{2\mu_g}\mathbf{b}^2. \tag{24}$$

We have used the following abbreviations:

$$\varepsilon_g = \frac{1}{4\pi G} \text{ and } \mu_g = \frac{4\pi G}{c^2}.$$

We have written the mechanical stresses [Eqs. (17)–(24)] separately for a purely electric (or gravistatic) field and a purely magnetic (or gravinetic) field in both the direction of the field lines and transversely to them. These expressions follow as special cases from the stress tensor given in Appendix B.

Merely considering the signs of the various quantities reveals some interesting qualitative differences between EM and GEM. In Sec. III, we examine the consequences of the opposite sign of the mechanical stresses and the energy density and then consider the energy current density distribution in some examples.

III. CONSEQUENCES

A. Mechanical stress in the gravitational field: Momentum current density

When treating gravitation we usually emphasize that an essential difference between gravitation and electricity is that while there are "two kinds" of electricity, there is only one "kind" of mass, which we take as positive. Positive masses attract each other while like electrical charges repel each other. However, these categories of attraction and repulsion imply action-at-a-distance physics, whereas we want here to formulate the differences between gravitation and electricity via the properties of fields.

Equations (17)–(24) indicate us that the signs of the stress within gravistatic and gravinetic fields are opposite to those of electromagnetic fields, a fact originally pointed out by Heaviside.[3] In EM fields, there is tensile stress in the direction of the field lines and compressive stress orthogonal to the field lines;[13] in gravitational fields, there is compressive stress in the direction of the field lines for both the **g-** and

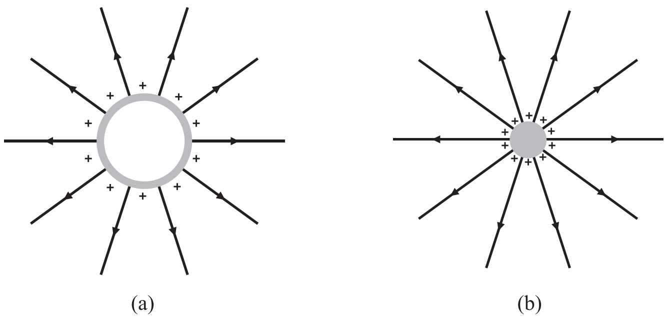

Fig. 1. (a) The electric field pulls outward on the sphere. (b) If the sphere is compressed while the charge is held constant, new field is created. For this, energy has to be supplied to the system.

b-fields and tensile stress transverse to them. Because mechanical stresses are equivalent to momentum current densities, we can say that for analogous arrangements of the sources of the fields, the flow direction of momentum currents in GEM fields is opposite to that in corresponding EM fields.[14] With this in mind, we consider the distribution of mechanical stress for a simple geometry, first in the electric case and then in the gravitational one.

Our system is a hollow sphere bearing a uniformly distributed positive charge as sketched in Fig. 1(a). The interior of the sphere is known to be free of field. If the sphere is made of an elastic material, it will inflate when charged. The traditional action-at-a-distance description is that the charges on the surface repel each other. A problem with such a description is that the straight lines connecting the "repulsing" parts of the surface have to run through the field-free interior. A physically better description would be as follows: "The field is under tensile stress in the direction of the field lines. The field lines end on the surface of the sphere and the tension is passed on to the sphere, with the result that the electric field pulls outward on the sphere."

Now for the gravitational analog. Consider a thin spherical shell of mass m_0 as sketched in Fig. 2(a). As in the electric example, its interior is field-free. Since positive masses attract each other, the shell would shrink, but it is prevented from doing so because it is made of solid material. In terms of fields, we should regard the gravitational field as pushing on the shell from the outside.

Similar statements can be made for the **b**-field. Instead of a hollow sphere, consider a hollow cylinder or a tube moving longitudinally. The interior of the tube is field-free. Using the "Lorentz force" equation, we can conclude that the different parts of the tube lying on one circumference "repel"

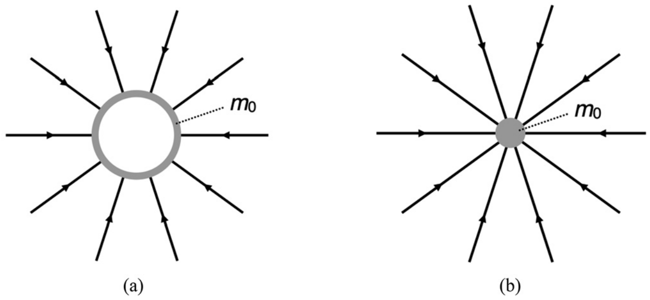

Fig. 2. (a) The gravistatic field pushes on the sphere. (b) If the sphere is allowed to shrink with the mass kept constant, new field is created. For this, energy has to be removed from the system.

each other. In terms of fields, we can express this as follows: The gravinetic field pulls on the tube. It follows that the field is under tensile stress in the direction orthogonal to the field lines. Just as in the electromagnetic analog, however, the mechanical stresses in the **b**-field are smaller by a factor of v^2/c^2 than in the **g**-field, where v is the velocity associated with the mass flow.

B. Energy density

It is easy to see that the energy density in the gravistatic field is negative. Consider first the electromagnetic case, Fig. 1(a). If we imagine compressing the sphere while keeping the charge constant [Fig. 1(b)], the field at some point outside the original sphere does not change. But as the sphere shrinks, new field is created in space that was previously field-free. We had to supply energy to create this additional field, and this energy is stored in the newly generated field. The energy density in the electric field is given by Eq. (9).

Now apply the same reasoning to the sphere of mass in Fig. 2. As the sphere shrinks, there is again a region of space which was field-free but is now occupied by a field. However, we did not have to supply energy to shrink the sphere; rather, we gained energy. If we stick to the idea that we can always specify an energy density, we must conclude that the field has a negative energy density, Eq. (11). These considerations extend to the **b**-field, but again the values involved are very small.

C. Energy current density

Whenever a body moves up or down in the gravitational field of the Earth, it exchanges energy with the field so-called potential energy. During such an exchange, an energy current flows within the field, and this can be calculated with Eq. (14). While **b** enters quadratically into the mechanical stresses and thus into the forces, it appears to the first power in the formulae for the energy current density, Eq. (14). As a consequence, the **b**-field manifests itself much more clearly in energy currents than in forces (momentum currents).

Figure 3 illustrates a simple example, which was proposed by Krumm and Bedford.[4] A long rod of mass m is pulled upwards, perpendicular to the Earth's surface.

Energy is supplied, which enters the rod from above. The energy flow (power) at the top of the rod is calculated according to the following classical formula:

$$P = \mathbf{v} \cdot \mathbf{F}. \tag{25}$$

Here, $\mathbf{v}$ is the velocity of the rod and $\mathbf{F}$ is the gravitational force whose magnitude is

$$F = mg. \tag{26}$$

As the force within the rod decreases from top to bottom, the energy flow also decreases. We ask for the decrease ΔP of the energy flow in a small height interval Δh, see Fig. 3(a). ΔP is that part of the energy flow which is transferred from the matter of the rod to the gravitational field within this interval. We first write the force as a function of the height h:

$$F = m(h)g = \rho_m A h g, \tag{27}$$

where ρ_m is the mass density of the rod. Thus, the energy flow through a sectional plane at the height h is

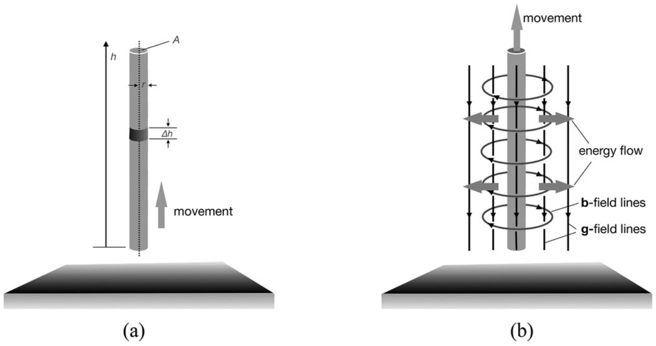

Fig. 3. (a) A rod is pulled from the top. In the process, energy flows within the rod from top to bottom. (b) This energy gradually leaves the rod sideways.

$$P = vF = v\rho_m Ahg. \tag{28}$$

Since the mass flow is

$$I_m = v\rho_m A, \tag{29}$$

the energy flow can be written as

$$P = I_m gh. \tag{30}$$

We are interested in the decrease ΔP of the energy flow in the height interval Δh:

$$\Delta P = I_m g\Delta h. \tag{31}$$

This energy flow exits the rod into the gravitational field.

We now ask what our GEM equations predict for this energy flow. We calculate the energy flow $\Delta P'$ in the gravitational field exiting the rod through the cylindrical shell of height Δh:

$$\Delta P' = 2\pi r\Delta hS. \tag{32}$$

To determine the energy current density S, we need the gravistatic and the gravinetic field. The **g**-field of the rod is so weak compared to that of the Earth that we need not consider it. The **b**-field is easily obtained from Eq. (8). Its magnitude is

$$b(r) = \mu_g \frac{I_m}{2\pi r}. \tag{33}$$

Since **g** and **b** are perpendicular to each other, we get the magnitude of **S** from Eq. (14) as

$$S = \frac{gb}{\mu_g}. \tag{34}$$

With Eq. (33), S becomes

$$S = \frac{gI_m}{2\pi r}. \tag{35}$$

Inserting S into Eq. (32), we get

$$\Delta P' = I_m g\Delta h, \tag{36}$$

which agrees with Eq. (31).

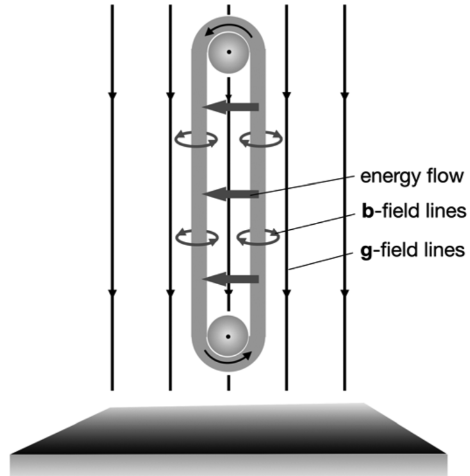

Fig. 4. In the left part of the rope, the energy flows within the rope upward, in the right part downward. In the process, its current strength increases upwards in the left part and decreases downwards in the right part. The circuit is closed by the gravitational field in which the energy flows from the right to the left part of the rope.

Thus, Heaviside's theory consistently describes our local energy balance: The energy flowing away from the rod via the field is equal to that flowing into the rod from above.

This result also shows that there cannot be a factor 4 in the equation for the energy flow, as is the case in the Thirring-type representations of GEM. Apparently, the Heaviside and Thirring theories correspond to two different mappings of EM onto gravitation.

Up to this point, we have calculated only the energy current density near the surface of the rod. The question of the total current density distribution is more difficult because it has sinks in the field and the current density distribution extends out into areas far from the rod. To circumvent this problem, we consider a different example, that of a heavy rope running over two pulleys. This is sketched in Figs. 4 and 5. The rope can equivalently be imagined as water flowing in a pipe.

To determine the path of the energy within the gravitational field, we need the **g**- and **b**-fields. We again neglect the contribution due to the rope; the **g**-field is simply that of the Earth. The **b**-field is an old acquaintance, having the same form as that of two parallel wires carrying oppositely directed currents; see Fig. 6. The vector multiplication $\mathbf{g} \times \mathbf{b}$

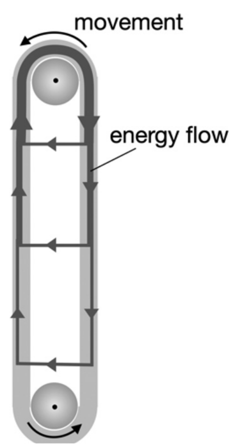

Fig. 5. Energy flow corresponding to Fig. 4, schematically.

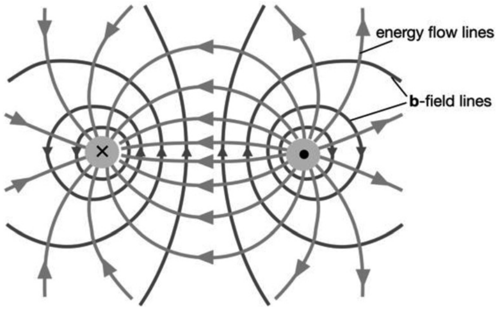

Fig. 6. Horizontal section through the arrangement of Fig. 4. The **g**-field lines point into the plane of the drawing. From the right to the left section of the rope, the energy flows through the gravitational field.

is straightforward, and the streamlines of the energy flow are orthogonal to the **b**-field lines. Energy transport is largely localized to the vicinity of the two sections of the rope and is a large effect, well known from daily life.

In setting up this arrangement, note that two conditions have been met: (1) The energy flux distribution has no sources or sinks within the gravitational field, and (2) each energy streamline runs in a plane, so the distribution can be represented in a cross-sectional plane. Both conditions are automatically satisfied for any closed mass flow such as an arbitrarily-shaped water circuit or a rotating vertical wheel.

IV. CONCLUSION

In order to avoid action-at-a-distance without using general relativity, one can describe classical gravity with fields. To do so, it is necessary to introduce the gravinetic field **b** in addition to the usual **g**-field. This results in a theory, gravitoelectromagnetism, which is analogous to Maxwell's electromagnetic theory. A striking and consequential difference between EM and GEM involves some reversed signs. The energy densities of both the **g**- and **b**-fields are negative, and the mechanical stresses in the GEM field are reversed with respect to those in the EM field: Compression becomes tension and vice-versa. Using energy density and energy flux density, the local energy balance for the gravitational field can be formulated in the context of gravitoelectromagnetism. In particular, the concept of potential energy can be replaced with a modern concept of energy stored in fields.

APPENDIX A: GRAVITATIONAL LORENTZ FORCE ON AN AIRPLANE

Electromagnetic forces are much stronger than gravitational forces; for example, the electrical repulsion between two protons is some 10^{37} times stronger than their gravitational attraction. Easily observable gravitational forces result only for bodies with very large mass, such as the Earth. In addition, gravinetic forces are typically smaller than gravistatic forces by a factor $v_1 \cdot v_2/c^2$, where v_1 and v_2 are the velocities of the interacting bodies.

To get an idea of the magnitude of gravinetic forces, consider an airplane of 400 tonnes (400 000 kg) flying eastward above the equator. The surface of the Earth is also moving in the eastward direction. So we have what can be compared to two electric currents flowing parallel to each other in the same direction. The corresponding conductors or charge carriers are known to attract each other.

To solve the plane problem, we would have to calculate the gravinetic field of a rotating sphere but that is mathematically quite complicated. However, since we are only interested in the order of magnitude of the effect, we map this problem onto the geometry of a charge moving parallel to the axis of a current-carrying wire. We replace the rotating Earth with a massive cylinder 1000 km in diameter moving along its symmetry axis. To calculate the mass flow, we also need the density of the cylinder, which we take to be $\rho = 4 \times 10^3\,\mathrm{kg/m^3}$. As the speed, we choose the speed of the surface of the Earth, i.e., about 1700 km/h. The airplane flies at speed 1000 km/h relative to the Earth, so as "seen from outside", i.e., in an inertial frame in which the center of the Earth is at rest, the velocity of the plane is 2700 km/h = 750 m/s. We assume a flight altitude of 10 km.

We obtain a mass current I_m of the passing cylinder of about 1.5×10^{18} kg/s. Setting $r = 5.1 \times 10^5$ m and $\mu_g = 4\pi G/c^2$ in Eq. (33) gives a **b**-field strength of $\sim 4.3 \times 10^{-15}\,\mathrm{s^{-1}}$. With $v = 750$ m/s, the magnitude of the corresponding force from Eq. (16) is then $F = mvb \sim 1.3 \times 10^{-6}$ Newtons. The direction of the force is away from the cylinder, effectively lightening the plane by about 0.1 mg.

APPENDIX B: THE MAXWELL STRESS TENSOR AND ITS GEM ANALOG

The expressions given in the text for the compressive and tensile stresses in the electromagnetic field can be summarized in a single tensor equation, the well-known Maxwell stress tensor:

$$T_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \left(\frac{\varepsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \right) \delta_{ij.} \tag{B1}$$

The analog gravitoelectromagnetic tensor is

$$T_{ij} = -\varepsilon_g g_i g_j - \frac{1}{\mu_g} b_i b_j + \left(\frac{\varepsilon_g}{2} \mathbf{g}^2 + \frac{1}{2\mu_g} \mathbf{b}^2 \right) \delta_{ij}. \tag{B2}$$

A comparison of Eqs. (B1) and (B2) shows that compressive and tensile stresses are swapped. From Eq. (B2), one can directly read our Eqs. (19) and (20) as well as (23) and (24) and, just as in the electromagnetic case, derive the Lorentz force equation.

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