Proceedings of the Singapore National Academy of Science, Vol. 15, No. 2 (2021) 135–150

DOI: 10.1142/S2591722621400123

Creativity in Action: Exploring Cross-Cutting Models of Self-Organization and Emergence in Science and Technology

Manfred Euler
IPN - Leibniz Institute for Science and Mathematics Education
Olshausenstrasse 62, D-24118 Kiel, Germany
euler@ipn.uni-kiel.de

Published 7 July 2021

This review presents a sequence of exemplary experience-based encounters with self-organizing systems on different levels of difficulty. Based on hands-on experiments and creative modeling it provides a viable educational road to build up a deeper understanding of self-organization principles and their comprehensive nature. Theories of self-organization describe how patterns, structures and new types of behavior emerge in energetically open systems, resulting from the local interaction of many components. As an external control instance is missing, the underlying philosophy is counterintuitive to our habits of causal thinking. This thematic and conceptual framework impacts on many STEM domains and presents a blueprint for modeling emergent structures and complex functions in natural and technological systems. It reveals unifying principles that can help in reducing, in structuring and, finally, in understanding and controlling the emerging complexity. An overview across diverse STEM domains highlights the role of this overarching concept. This cross-disciplinary approach can help in improving the dialogue and the knowledge exchange between the individual fields. Moreover, in a self-referential fashion, the modeling of self-organization provides us with fresh perspectives to reflect our own creative processes.

Keywords: Science education; STEM literacy; experience-based learning; creativity; modeling; self-sustained oscillations; synchronization; self-organization; coherence, model-order reduction; emergence; universality; smart systems.

1. In Touch with Universal Self-Organization: A Project-Based Approach

Education is challenged to improve science literacy and to keep pace with the highly dynamic evolution of scientific and technological knowledge. General methodological considerations how to teach STEM-subjects are widely addressed in educational strategies. In contrast, the question which subjects should be taught and which conceptual tools are suited to foster excellent and creative young minds has attracted much less attention. On the one hand, the progression of knowledge creates an increasing spectrum of highly specialized domains. Content specific educational implications are difficult to deduce from that rapid development. On the other hand, from a meta-perspective, the evolution of new structures and functions in a multitude of systems across all domains of experience reveals unifying principles. Theories of self-organization address this issue. They describe how patterns, structures and new types of behavior emerge in systems that consist of many components, resulting from interactions fully internal to the system. They provide a conceptual perspective on unity in diversity with promising educational implication and present a fruitful guiding line to select content and methods that satisfies subject specific as well as more general epistemological demands. The underlying ideas of modeling interactions in dynamical systems and resulting implications are relevant for addressing present-day as well as foreseeable future challenges. They can help in reducing, in structuring and, finally, in understanding and controlling the emerging complexity in different domains.

Self-organization applies to a broad class of processes in the inanimate as well as in the animate world, including perception, action and cognition. Moreover, many technologies depend on that principle in one way or another. The generic traits are remarkable. Largely irrespective of the material substrate and the specific interaction mechanisms, the resulting evolutionary patterns show qualitative and, in some cases, even quantitative similarities. In spite of its overarching nature, the idea of autonomous and even smart processes is difficult to comprehend intuitively for several reasons. The absence of an external supervising authority is in conflict with our self-conception of intelligent agents, seemingly in full control of voluntary actions. It also runs counter the intuitive notions of unidirectional, nonholistic cause-effect relations. More technically, the self-structuring processes depend on nonlinearities that pose additional conceptual and mathematical difficulties. All this calls for comprehensive educational efforts that address these issues.

Along that line, this review presents a viable road to approach the comprehensive, cross-disciplinary nature of self-organization in STEM subjects starting from basic physical models. It follows the educational strategy to foster creative modeling among students by suitably interweaving procedural and conceptual knowledge, developed in a companion article [1]. Students actively engage in understanding complex phenomena by constructing scientific models to describe, explain, predict, and to control phenomena. The focus on improving the interlinking of factual and procedural knowledge is the core idea in modeling theories of physics instruction [2, 3]. The ability to devise, use, test, and expand or revise models is generic and represents a set of crucial skills throughout the different STEM domains, although domain specific views set the focus on specific skills somewhat differently [4, 5]. Creative modeling is the generative core of scientific methodology [6], far from being beyond the scientific method, as suggested by some educators [7].

Self-organization can be considered a principle theory that sets out a generic framework for the evolution of structures and functions. We describe projects using dynamical systems as basic paradigms of self-organization. The systems are appropriately simple to ease the creation of conceptual models and to refine and extend them to mathematical models that permit quantitative predictions. At the same time, they are sufficiently complex to embody relevant principles and to serve as approximations of self-structuring processes in more distant and less accessible fields. As epistemic mediators, they bridge the gap towards forming more explicit domain specific theories which interlink the basic conceptual entities of the respective domain. This renders educational reconstruction more demanding, because knowledge from the specific level of organization must be integrated to successfully unfold this generic approach. Additionally, reflections are required to address communalities but also specific differences between the domains.

Starting from simple hands-on experiments, the article elaborates basic principles and universal products of self-organization in the framework of coupled self-sustained oscillators with a focus on synchronization phenomena. Exemplarily, this knowledge is applied to acoustic information processing. Our acoustic sense is almost predestined for exploring the constructive role of self-organization from the perspective of internal observers. Moreover, an analysis of what the ear communicates to the brain provides us with valuable insights into the ways how our brains create representations of the external world. A more advanced conceptual section translates the hands-on experience of exploring synchronization in terms nonlinear dynamical models. Although the models are derived from a fully classical context, the models foster a basic understanding of analogous organizational principles in the quantum domain and their applications in diverse cutting-edge technologies, on which our societies depend. Future-oriented STEM education requires a clear focusing on these aspects which are largely underrepresented in the present educational mainstream. A more general outlook presents an overview on the relevance of self-organization across a broad spectrum of STEM-subjects. This comprehensive perspective underlines the educational relevance of the subject to sensitize creative young minds. Moreover, in a self-referential fashion, it provides us with fresh perspectives to reflect our own creative processes.

2. Interacting Self-Sustained Oscillators: Towards a Basic Model of Dynamical Self-Organization

Self-sustained oscillations show up in many domains of experience. Their persistent periodical or quasi-periodical behavior depends on suitably tapping an energy source to sustain the dynamics. Everyday examples include the squealing of brakes, the ringing of doorbells, the sounds of musical instruments such as bowed strings, flutes or organ pipes. More complex self-oscillations and rhythmic processes abound in biological systems, e.g., they underlie breathing, heartbeats, neural oscillations and metabolic cycles. The study of self-oscillations is a promising educational context to develop a deeper understanding of the inherent regulatory principles. Moreover, the coupling of self-sustained oscillations provides a basic model to develop a comprehensive framework of dynamical self-organization [8]. Coupling leads to the emergence of new, widely generic behavioral patterns and includes transitions from unordered states to order and coherence in a wide range of natural and designed systems.

The feedback cycle of sensing the state of a dynamical system and of feeding back energy with the proper phase is the standard model for designing self-oscillations. Jenkins [9] gives a basic review of this general principle, which is independent of the respective material realization. Based on earlier work on various models of self-organization [10, 11] a learning process study was carried out. It investigates the educational suitability of these systems to gain insight into basic principles via largely self-regulated learning [12]. School students ($N = 38$, grade 11-12) cooperating in groups of two took part. The study investigated the students' creative model construction starting from exploring simple self-oscillating systems such as the friction pendulum or the woodpecker toy.

The groups developed graphical models of the interactions and the flow of actions that underlie the observed dynamical behavior. The individual modeling processes were interspersed by phases of comparing and evaluating the models between the groups. Additionally, they were enriched by reflections, stimulated by the instructor. Along these cyclic cooperative activities, the students successfully constructed reasonable models of the interaction cycles (Fig. 1). They addressed the relevant and more abstract conceptual ideas such as feedback and dynamical equilibrium. Follow-up tests and

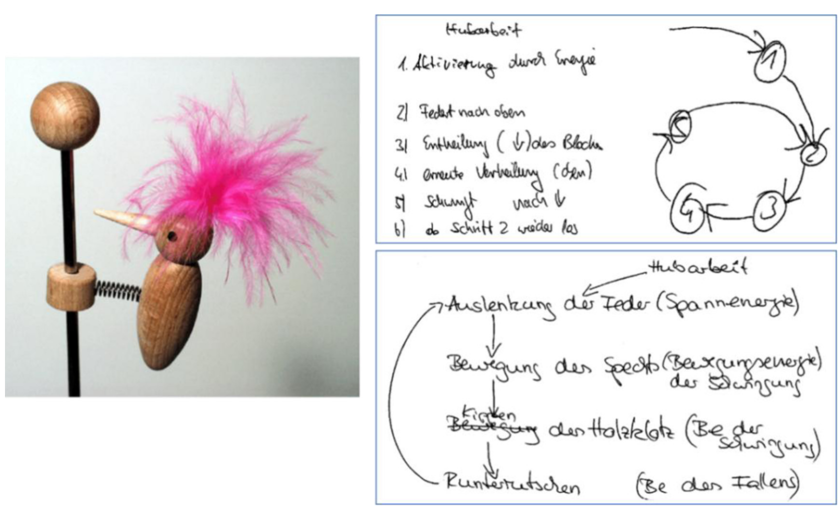

Fig. 1. The woodpecker as a toy model of self-oscillations. Two graphs, developed by different student groups, depict models of inherent feedback cycles. The cyclic flowcharts show largely adequate descriptions of the periodic stick-slip motion along the vertical pole. The locking and unlocking of motion depends on the tilt angle of the sleeve, to which the woodpecker's body is coupled elastically. The second diagram also includes energetic descriptions (in parentheses).

interviews 4 months later revealed the important role of graphical modeling for the students' retention and their recall of the functional principles and abstractions. In the interviews, the students presented examples how they used their experience in the meantime to devise models in other STEM subjects and to identify regulatory loops. The examples range from temperature control at home to earth systems and the greenhouse effect. This wide spectrum provides a clear evidence of cross-domain transfer, initiated by the modeling processes of self-sustained oscillations.

The single oscillator model can be extended and generalized. By including coupling effects between two or more self-sustaining oscillators these systems provide the substrate for dynamical self-organization and coherence. Any implementation of suitably coupled systems with autonomous cyclic functionality will show the effects. The present experiment uses metronomes to explore the basics. This was the first design to present a detailed investigation of the resulting coupling products [13]. Several free-running slightly detuned metronomes produce an irregular sequence of clicks according to their natural frequencies. A one to all interaction is implemented by placing the metronomes on an oscillating base plate. The plate is part of a larger pendulum arrangement with the plane of swing in parallel with the swinging metronome bobs. Initially, the superposition of the individual beats sounds like an unordered mess. All of a sudden, the metronomes conjure up to an ordered behavior and tick in unison in spite of slightly different internal settings (Fig. 2). Listening to the incoherent concert of clicks and their sudden synchronization is an intriguing experience. The interacting systems agree on a common tempo and adjust their timing autonomously without external regulatory input. The whole array acts coherently. Most listeners feel challenged to come up with ideas about possible mechanisms and to discuss relations to other domains. Very often the experiment was seen as a social metaphor, e.g., demonstrating the switching of opinion, induced by a strong influencer, or the transition to collective behavior, such as the emergence of rhythmic hand clapping during frenetic applause.

Modeling the synchronization effect is more demanding than pure self-oscillations on which it builds up. It requires the inclusion of an additional regulatory level that acts back on the timing of the individual oscillators by modulating their phase. In the present case of mechanical clocks this feedback loop is fully transparent and literally tangible. Each metronome is steered by a phase-controlled energy input. It provides accelerating kicks along with the audible clicks during each zero crossing. Shaking an individual metronome shows that timing and beat rate can be modulated by periodical external forces. This is the explanatory core. The metronomes excite platform oscillations via reaction forces. The resulting collective motion acts back on each single system and changes its rate. Fast metronomes are slowed down, slow ones are speeded up. Depending on the degree of detuning and the coupling strength, a coherent state can evolve with the metronomes oscillating in phase.

The pointer model provides a helpful visual tool to describe this process. Rotating pointers indicate

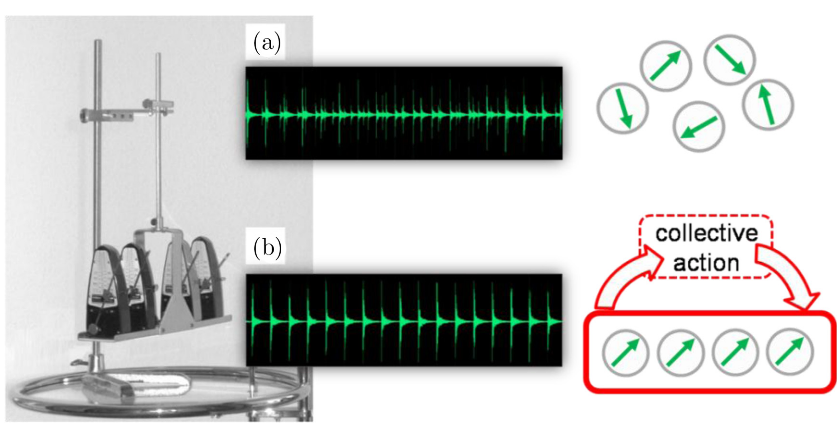

Fig. 2. Self-organization of coupled metronomes. (a) Click sequence of independent uncoupled metronomes. (b) Coherent clicks after coupling via base plate oscillations as a collective feedback.

the instantaneous phase of each oscillator. Initially, without coupling, they evolve independently with an individual rotation rate. Their directions are seemingly random (Fig. 2). After switching on the interaction, the phase pointers mutually adjust in response to the collective feedback via platform motion and finally rotate in synchrony. All phase arrows point to the same direction. Vector addition of the individual pointers demonstrates the resulting amplification effect. The combined action grows with the number N of the individual units. In the case of incoherent superposition, the average response is zero. The fluctuations around the average, i.e., the noise, grow as $N^{1/2}$ as it is the case in random walks. In addition to amplification, this feature of coherent processing greatly improves the signal to noise ratio. The qualitative explanation by the pointer model of oscillations is expandable and compatible with more advanced approaches. It provides the basis to devise quantitative mathematical models of the resulting dynamics.

The dependence of synchronization on detuning shows generic behavioral patterns. This is studied by examining the base oscillations from two interacting metronomes. Figure 3(a) shows the setup and a typical spectrum obtained from a stepwise detuning of metronome 2 while keeping the setting of metronome 1 fixed. For strongly detuned metronomes, two main frequencies f_1 and f_2 can be identified which deviate from the free running frequencies. Coupling results in a mutual 'attraction' or 'pulling' of the frequencies that depends on detuning. Additionally, further combination products show up. Their frequencies are given by a linear combination $f_c = mf_1 \pm nf_2$ (m, n small integers). The mutual pulling of the metronome modes increases with decreasing detuning. At a certain critical value, they merge to a single frequency $f_s = (f_1 + f_2)/2$. The oscillations are phase-locked. This synchronized state persists over a certain range of detuning. At a second critical frequency phase-locking breaks up.

3. The Sounds of Self-Organization: Exploring Universal Dynamical Products from an Inner Perspective

This section builds upon basic physics knowledge how oscillation and waves superpose. It extends these concepts by nonlinear interactions which are subsequently applied to model acoustic perception effects. This switch of focus is intriguing, as our acoustic sense opens up a new sensory channel to experience dynamical products of self-organization. Synchronization is a universal phenomenon largely independent particular feedback mechanisms and the dynamical substrate [14]. An analogous experiment with acoustic self-oscillations demonstrates the universality of the coupling scenario. Two tuneable organ pipes with different frequencies are placed in parallel with opposing mouths. Their oscillations interact due to the overlapping sound fields. The distance is selected such that the coupling is weak. The frequency spectra (Fig. 3(b)) show the attraction of the pipe modes f_1 and f_2. They fuse more or less abruptly to the synchronized

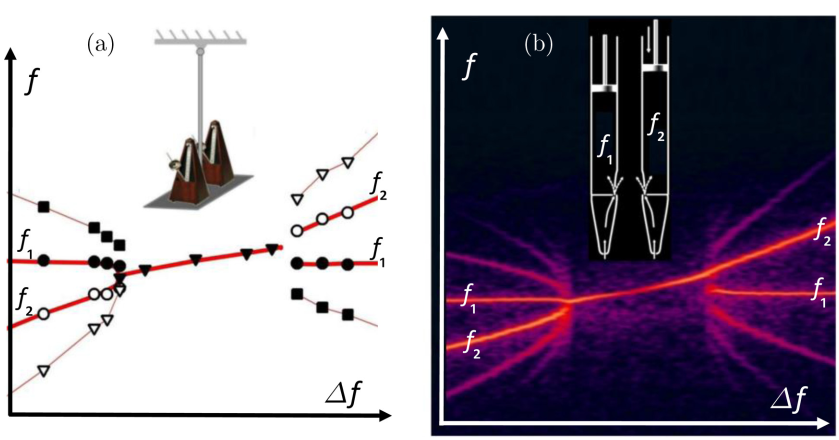

Fig. 3. (a) Frequency spectra of the base plate oscillations. Metronome 1 is kept at a fixed setting. The beat rate of metronome 2 is varied stepwise. (b) Sound spectra of 2 organ pipes coupled by their sound field (f_1 fixed, f_2 tuned upwards by reducing the resonator length, $\Delta f = f_2 - f_1$).

state depending on the detuning Δf. Different from the metronome experiment the fusion is not symmetric. As one pipe is slightly louder it pulls over and synchronizes the oscillations of the weaker partner.

The acoustic experiment allows us to perceive the resulting combination products. Although we cannot resolve the spectral details, the character of the beat-like sounds close to synchronisation is clearly different from beats that we perceive in the superposition of two tones with neighbouring frequencies. The periodical loudness variations are asymmetric and resemble a ratchet or a step-like process. We perceive kind of a slowly changing holding state followed by a sudden change. Outside the locking range the cubic combination tone $f_c = 2f_1 - f_2$ is the most prominent audible interaction product. Its pitch goes down upon increasing f_2. This experiment is helpful in identifying corresponding nonlinear products of self-organization that underlie our own acoustic perception.

Our acoustic sense is almost predestined for exploring the constructive role of self-organization. It provides us with an introspective window to tune into these processes. In contrast to the passive working of a microphone, the ear is an active system that amplifies signals mechanically [15]. The so-called cochlear amplifier is based on the interplay of two groups of sensory cells in the inner ear. They detect the oscillations of the basilar membrane, along which mechanical waves propagate in response to the incoming sound signals. One type of cells, the inner hair-cells, acts as mechanical sensors transforming the wave pattern to neural signals. A second group, the outer hair-cells, has an additional motor function and periodically feeds in energy. In order to operate properly, the sensory and motor processes must be part of a phase-sensitive regulatory circuit that ensures the proper timing of the energy input. Moreover, the system must be kept close to the threshold of self-excitations. In a technical comparison, the active ear works in a way similar to the regenerative receivers, known from the early days of radio engineering. While the latter implement electronic feedback, the powers of our auditory system are based on neuro-mechanical feedback. The regulatory circuit resulting from a cooperation of passive sensory and active motor processes is analogous to the feedback cycles in the above models. The metronome pendula and the jets in the organ pipe both sense oscillations and control the energy input analogous to the hair-cells in the cochlear amplifier. Accordingly, we should expect fully analogous dynamical products that result from the cooperative actions.

These products can be studied by listening to the combined action of two simultaneously presented pure sine tones. They generate a variety of audible combination products that depend on the levels and the frequency of the primary signals [16]. We can perceive them as periodical variations of loudness (beats), sensations of roughness, and new combination tones. Figure 4(a) schematically shows these percepts and their dependence on the frequency

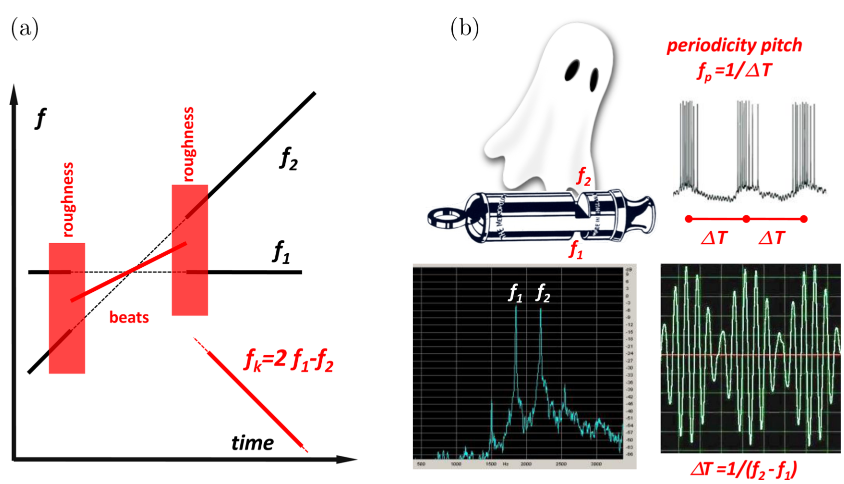

Fig. 4. (a) Two tone perception with beats and roughness and the cubic combination tone. In this experiment f_1 is constant and f_2 increases with time, (b) Virtual pitch of the 'ghostly' sound from a signal whistle with two separate acoustic oscillators. This percept originates from the periodicity of neural bursts, synchronized with the envelope of the beat signal.

difference of the primary signals [17]. It is common that systems respond linearly at low input level and gradually pass over to saturation at high levels. The ear also shows the latter type of limiting nonlinearity. Additionally, even for low signal levels, essential nonlinear effects persist due to the threshold behavior of the hair cell amplifier. The resulting cubic combination tone is most prominent and clearly audible on the high frequency side of the beat and roughness sensation. It is the dynamical signature of self-organization processes. It exists physically on the level of basilar membrane oscillation due to mechanical feedback. Moreover, it is measurable in the ear canal because part of the energy travels outwards through the transduction chain of the middle ear. These so-called otoacoustic emissions are the objective retroactions of neuro-mechanical feedback in the inner ear. They indicate the integrity of the cochlear amplifier and are used accordingly in medical diagnosis [18]. From a meta-perspective they can be considered the echoes of self-organization that act back on the outside world.

The next step in the auditory chain requires integrating further knowledge from biology. Neural spikes are the information carriers on the subsequent level of signal processing. The information can be encoded in the spike rate as well as in the phase of individual spikes. Although a single neuron fires according to the all-or-nothing principle, the superposition of phase modulated spikes is able to transmit the temporal structure of low frequency signals in real time. This occurs by bursts or volleys of neural activity similar to the phase-locked clicks of the metronomes. Some additional phase jitter smears the timing of the individual spikes. The perception of the alarming sound from a signal whistle depends on that process (Fig. 4(b)). The "Bobby whistle" consists of two single sound sources ($f_1 = 1.8\,\text{kHz}$, $f_2 = 2.2\,\text{kHz}$). When they are blown together, a loud alerting low pitch sound is heard. It has a somewhat 'ghostly' character, because there is no corresponding spectral component in the physical world outside. However, in our perception it is fully real and even more penetrating than the individual pipe signals.

This sound is a salient manifestation of periodicity pitch resulting from neural synchronization and its processing at subsequent levels of the auditory system up to the cortical level. The superposition of the two signals creates beats. The periodical modulations of the loudness depend on the frequency difference $\Delta f = f_2 - f_1$. The beats trigger volleys of largely synchronized neural spikes. According to the periodicity principle, the repetition rate of the neural bursts $\Delta T = 1/\Delta f$ is interpreted as pitch by our brains. In a similar way, our auditory system is able to reconstruct missing or masked spectral information in the low frequency range from high-frequency components (residual pitch perception). This is why we can reconstruct the fundamental frequencies of voices or alarming sounds even though they are hidden in ambient noise. For the same reason, opera singers train the high frequency formants of the singing voice in order to overcome masking by low frequency orchestral sounds. This presents an important process of gestalt perception or form completion. On the basic level, it depends on synchronization. The evolution of the auditory system uses these 'ghostly' products of self-organization in highly constructive ways. They ensure our survival by detecting alarming sounds in noisy environments. The very same process serves our musical pleasure.

To feel the acoustical world is anything but a passive mechanism. The interlocking of sensory and motor processes makes the inner ear a truly smart structure that adapts dynamically to the incoming signals. The self-organized activity of a dynamical filter system has a price in terms of full signal fidelity. It creates new interaction products. Additionally, the adaptive resonances both on the mechanical and the neural level of processing can suppress weak signal components in the neighborhood of more powerful signals. This shortcoming of self-organized perceptual coding is exploited technologically by data compression algorithms. Our electronic communication devices use them to reduce the information content of digital signal representations and to diminish the data transmission rate (cf. Jayant *et al.*,) [19].

This highly condensed overview on the biophysics of hearing demonstrates the truly trans-disciplinary impact of dynamical self-organization. It gives an introductory exposition to identify the common principles behind a variety of phenomena that underlie the self-orchestrated construction of our highly differentiated and complex acoustic experience. An analysis of what our ears communicate to the brain and how top-down mechanisms act back on more peripheral levels of processing provides us with valuable insights into the ways how our brains create representations of the external world.

This perspective on self-organization all the way up to the world of ideas and all the way down towards emerging actions will be resumed below.

4. On Pendula and Washboards: From Qualitative Explanations to Dynamical Models

While the preceding sections gave largely qualitative expositions, we present a way to transfer these to quantitative models. This sequence is more advanced. It addresses introductory physics courses at university level but parts can be broken down to upper secondary level, if a sufficient interlinking with mathematics is provided. Synchronization is an essentially nonlinear effect. It must be distinguished from resonance, where an oscillator passively follows an external driving signal. Resonance is a common topic in introductory physics, whereas synchronization is only sparsely covered. This neglect hinders a deeper understanding of this universal concept and its role as a powerful paradigm of self-organization. The phase pointer picture of emerging synchronization (Fig. 2) provides us with an intuition how to develop a basic mathematical model. The resulting differential equations can be solved explicitly or by computer simulations, using common table calculation tools such as EXCEL for carrying out the computations and for graphical representations of the emerging dynamics. This approach has the pedagogical advance of being completely transparent to the modelers. It depends on devising straightforward algorithms based on difference equations for numerically approximating differential equations [20]. Beyond formal manipulations of mathematical symbols, this has the benefit that students learn to understand what the differential equations express in a basic sense. Learners can develop a feeling for the expressions and an intuitive acquaintance for their operational meaning. This is important for guiding their knowledge progression. Modeling expertise critically depends on this flexible and generative understanding of mathematical tools (cf. Sherin [21] for basic examples).

As the subsequent mathematical models are based on angular frequency $\omega = 2\pi f$ we omit the adjective angular. The frequency can be time-dependent. Its instantaneous value is the phase rate $\omega(t) = d\varphi/dt$. For simplicity we consider the periodical driving of a single metronome with phase φ_i and frequency $\omega_i = d\varphi_i/dt$ by an external signal with a constant frequency $\omega_e = d\varphi_e/dt$. Without coupling, the drift of the relative phase $\varphi = \varphi_i - \varphi_e$ is a linear function of time. Its rate $d\phi/dt$ is constant and corresponds to detuning $\Delta\omega = \omega_e - \omega_i$. Coupling introduces an additional phase modulation that changes the instantaneous frequency periodically. A sinusoidal dependence on relative phase is an obvious choice in accordance with the phase pointer picture in Fig. 2. Additionally, the coupling strength must be specified. It can be expressed by a constant that turns out to be the critical frequency difference $\Delta\omega_c$ for breaking up the synchronized state. Thus, the phase rate can be written as

$$\frac{d\varphi}{dt} = \Delta\omega - \Delta\omega_c \sin\varphi. \tag{1}$$

This equation is named after Adler [22], who derived it for locking phenomena in driven electronic auto-oscillations. It was generalized to describe ensembles of phase oscillators in chemical and biological systems [23]. This ordinary nonlinear differential equation reduces the behavior of driven or coupled self-sustained oscillations to the first order dynamics of a single-phase variable. The Adler equation depends on approximations valid in the limit of weak coupling [24]. After variable separation, it can be integrated by using the tangent half-angle substitution [25]. We skip the somewhat lengthy result and present the dependence of the time averaged mean frequency $\bar{\omega}_B$ on $\Delta\omega$ and the critical detuning $\Delta\omega_c$

$$\bar{\omega}_B = \sqrt{\Delta\omega^2 - \Delta\omega_c^2}. \tag{2}$$

Figure 5 shows the dependence on relative detuning $\Delta\omega/\Delta\omega_c$. Below critical detuning the oscillations are locked to the drive signal. There are no beats. The synchronized state breaks up at $\Delta\omega/\Delta\omega_c = \pm 1$. As the drive frequency is fixed, the modulation is only one-sided, different from the symmetric modulation spectra of 2 coupled oscillators in the preceding experiments. The onset of beats corresponds to a second order phase transition. This type of smooth change can be identified for negative detuning on the left side of the coupling spectra of Fig. 3. The upper transition in these experiments is not smooth. The resulting jump corresponds to a first order transition. Here, the weak coupling approximation of Eq. (1) breaks down and modeling must include additional amplitude modulation effects. The time dependence of the

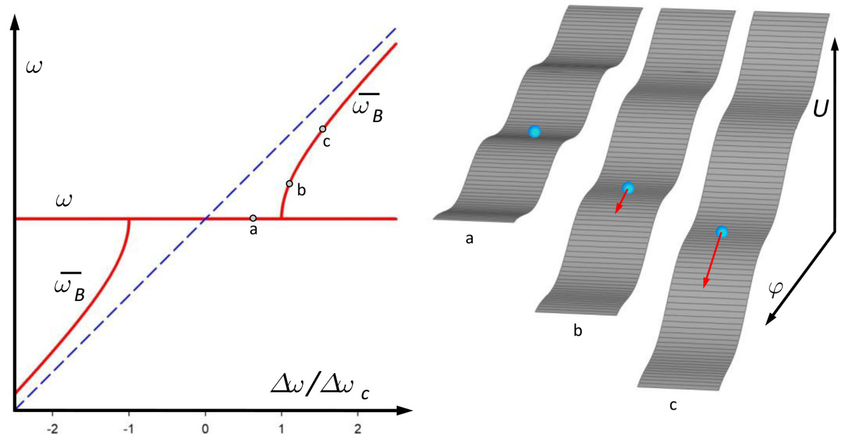

Fig. 5. Dependence of mean frequency on detuning. The tilted washboard model is shown for three values of detuning.

instantaneous frequency can be obtained readily from integrating the Adler equation numerically by table calculation programs or by the explicit formula

$$\frac{d\varphi}{dt} = \bar{\omega}_B^2/(\Delta\omega + \Delta\omega_c \cos(\bar{\omega}_B t)). \tag{3}$$

The instantaneous frequency varies periodically between $\omega_{\max} = \Delta\omega - \Delta\omega_C$ and $\omega_{\min} = \Delta\omega + \Delta\omega_C$ with the period $T = 2\pi/\bar{\omega}_B$. The time average is the geometric mean $\bar{\omega}_B = \sqrt{\omega_{\max} \cdot \omega_{\min}}$.

A simple mechanical model facilitates a qualitative understanding of the dynamics. A torque driven pendulum rotating in a highly viscous medium is an exact analogue. Viscous friction damps inertial effects. Therefore, in this over-damped case, the angular velocity is proportional to the total torque. A more practical design replaces viscous drag forces by eddy current braking [26]. The constant torque of the drive corresponds to detuning. In order to initiate periodical motion, it must overcome the maximum torque by the pendulum bob in horizontal position. The tilted washboard model provides an equivalent description from an energetic perspective. A fictive phase particle under viscous drag slides along an inclined track with periodical bumps. This shape corresponds to a potential energy landscape $U(\varphi) = -\Delta\omega \cdot \varphi + \Delta\omega_C \cdot \cos\varphi$. The negative derivative is equal to the phase rate from equation (1). If the washboard is only slightly tilted, the phase particle is trapped in a potential minimum. This is the phase-locked state. If the inclination exceeds a critical level, synchronization breaks up. The phase 'particle' starts sliding downhill. The phase rate changes periodically due to the periodical variations of the local inclination (Fig. 5). The time dependence is given by (3).

The phase rate is modulated by periodical undulations of the total driving force. Close to the threshold it stays practically constant over a longer period, followed by a rapid phase slip of 2π. Thus, the time course of the instantaneous frequency ω_B shows a periodical sequence of spikes (Fig. 6(b)). The phase modulation gives rise to new frequency components. They are whole-numbered multiples (harmonics) of the mean frequency $\bar{\omega}_B$ and correspond to the sidebands known from FM radio. In the present case of fixed drive frequency, the harmonics are one-sided. Close to the threshold of synchronization the spectrum includes many harmonics (Fig. 6(c)). With stronger detuning the relative contribution of the washboard ripples decreases. Thus, $\omega_B(t)$ becomes more sinusoidal and higher order harmonics gradually fade.

Such a phase modulated acoustic signal produces an auditory sensation highly similar to the beats and combination products shown in Fig. 4(a). Therefore, one can consider $\omega_B(t)$ kind of a nonlinear beat signal. Different from a linear superposition of two periodic signals, the beats do not reside in amplitude but in phase modulations. In spite of the approximate nature of the phase model it captures essentials of the complex dynamics of self-organization processes and the resulting behavioral patterns in a wide spectrum of natural and designed dynamical systems. This indicates the amazing effectiveness of model-order reduction on which equation (1) is based.

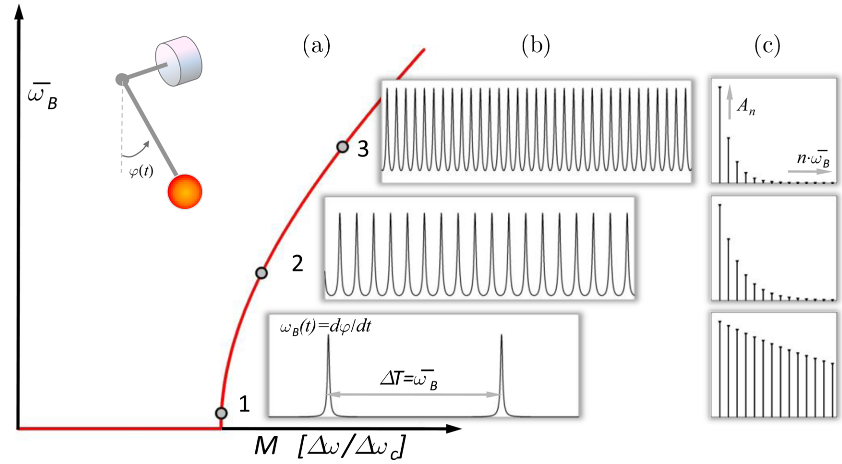

Fig. 6. (a) Mean frequency of the over-damped pendulum as a function of the driving torque M. (b) $\omega_B(t)$ for three different overcritical values of M (equivalent to $\Delta\omega/\Delta\omega_c$): (1) M = 1.001, (2) M = 1.1, (3) M = 1.3. (c) Corresponding amplitude spectra $A_n = f(n\bar{\omega}_B)$ normalized to A_1.

5. The Versatile Toolbox of Self-Organization: Engineering Coherence in Quantum Technologies

The driven pendulum and the equivalent washboard model of synchronization present a conceptual bridge towards relevant organizing principles of quantum technologies that orchestrate quantum coherence. The phase model emerged from considering coupled auto-oscillations. Although these systems belong to the classical macroscopic world, they can also assist in establishing a correspondence with related phase coupling processes on the quantum level where, often enough, our classical intuition fails. Numerous technological products depend on auto-organization of quantum processes and on engineering quantum coherence. Future-oriented STEM education should provide sufficient space to address the productive potential of these technologies adequately. The above models are helpful in providing an introductory analogy-based approach on which furthermore refined elaborations can build up.

The most prominent example is the creation of coherent radiation via stimulated emission of light in lasers. In the research direction of synergetics the laser served as a powerful paradigm of self-organization [27]. The full laser theory requires the framework of quantum electrodynamics, which is beyond the scope of an introductory explanation. An initial classical intuition on emerging coherence is obtained from the phase pointer model in Fig. 2. It includes feedback induced by the resulting field in the resonating laser cavity and depicts the emerging coherence from the synchronized action of individual oscillators, upon which the field acts. The laser requires pumping by an external energy source. It is an example of self-organization in a non-equilibrium system: If the pumping power exceeds a critical threshold the laser action sets in. The coherent superposition of light quanta creates macroscopically observable interference effects as we know from experiments with laser pointers. In spite of macroscopic observability, we have to mind the underlying quantum nature and to express a caveat with respect to a naïve use of classical pictures. In the above model the number of oscillators and the phase are both well defined. This differs from coherent radiation fields: The lower the uncertainty of the phase, the higher the uncertainty of the photon number. This is the number-phase version of the Heisenberg uncertainty principle valid for coherent quantum states.

Injection locking in lasers is another important phenomenon that can be described semiclassically by the above phase pendulum. Light from a 'master' laser is coupled to a 'slave' laser. Properties of the former such as frequency stability are transferred to the latter by phase-locking [28]. Comb-like spectra with equidistant lines analogous to the spectra in Fig. 6 emerge. The so-called optical frequency combs are used as precision 'gears' to divide the oscillations down so that they can be counted electronically [29]. This principle opened up new technological roads to create measuring systems with extreme precision. New applications in high-rate optical data transmission are under development. Lasers have become a tool that revolutionized science and society. Many laser-based systems can be found at our homes. The present technological world is unthinkable without lasers and further systems that depend on orchestrated quantum effects. In particular, the underlying quantum coherence is a cornerstone of high-performance information and communication systems. Its importance for future technologies will even continue to expand.

Further examples of macroscopic quantum coherence include superconductivity, superfluidity and Bose-Einstein condensation. These are low temperature self-organization phenomena that emerge from the coherence of matter waves. Different from the laser, they occur in thermal equilibrium. We mention in passing the phase modulation and phase coupling phenomena that arise when two condensates are brought into a weak contact. They are known as DC and AC Josephson effects [30]. In superconducting systems these effects are applied in voltage standards and in high sensitivity magnetic field sensor systems. One can expect that systems based on coherent matter waves will become even more important in future technological developments. Again, the over-damped pendulum and the washboard model can be used to provide a classical intuition of relevant effects [31].

With an appropriate adjustment of the correspondences between the classical and the quantum variables such concrete models are extremely powerful. They can be considered engines of intuition that enhance our powers in creating new knowledge. Acting as epistemic mediators they reduce the epistemic distance to the quantum domain and facilitate a broader and more concrete understanding of abstract or complex phenomena, ready to unfold productively into new directions. With respect to the interdependence between science and technology such concrete representations are indispensible to promote conceptual ideas in designing technological applications. Strengthening these engines of creativity and innovation should become a central guiding line for future-oriented STEM education across the educational chain.

6. From Complex Self-Organizing Matter to Creative Ideas: An Evolutionary Perspective on Creative Processes

Beyond the foregoing perspective on physics-based technologies, ideas and concepts of self-organization and the evolution of complex adaptive behavior impact on many domains from chemistry to biology, from engineering to medical application, from neural to social networks, from small scale interactions to global networks. We give a few representative examples with selected references of this rapidly developing field:

- Emergence and design of complex structures and functions in supramolecular chemistry and in biochemistry [32]
- Self-organized dynamics and nonlinear control in biological systems [33, 34]
- Synchronization and self-organization from a network perspective [35]
- Artificial intelligence and learning algorithms for supervised and self-organized learning, dealing with big data [36]
- Self-organization in engineering and robotics [37]
- Basic models of brain and behavioral dynamics [38], models of rhythmic activity in the brain and the role of their couplings [39], adaptive resonance approaches to conscious experience [40].

The location of research and development at the intersection of different disciplines and the wide scope of the concepts is a challenge to traditional subject-focused education. Yet, the inherent comprehensive perspective underlines the educational benefits of addressing these subjects in suitable ways, in order to motivate and attract young creative minds to become engaged in developing their skills and competencies on a cross-cutting field. In the long run, this has the great potential for enhancing the dialogue, the exchange of ideas and the cooperation between different domain specialists. Core challenges of sustainable development in our societies and economies critically depend on the ability to bring together expertise from different domains in productive and innovative teams in order to counteract the imminent global threats.

Beyond the engineering perspective of understanding, predicting, and controlling the emerging behavior of complex systems, the self-orchestrated coherence in biological systems has important consequences for our own self-concept. It culminates in the question, also known as the hard problem of neuroscience [41]: How can the interaction of matter and fields in our brains turn into imagination and consciousness? Is our creative mind an emerging dynamical product of coherent brain activity, or is it something different depending on interventions from the outside? These ultimate questions have many possible answers that depend on the particular personal and cultural background. In evaluating potential answers, it may be helpful to refer back to Leibniz, who used the metaphor of synchronized clocks to discuss the relation of mind and matter. As he considered both entities as fundamentally different and non-interacting, he conceived the doctrine of pre-established harmony as an initial divine intervention which creates persisting coherence.

The preceding examples demonstrate that systems of interacting clocks or coupled self-sustained oscillators represent more than a mere philosophical metaphor. Beyond qualitative illustrations, these dynamic models embody universal organizing principles that underlie the emergence of new collective behavioral patterns. In many cases the emergent properties can be expressed by new effective laws, which are specific for the particular level of organization and the domain of experience. In quantum physics, similar organization principles can be used to orchestrate quantum coherence for creating innovative applications. In a biological context, the coupled clock model demonstrates within a fully mechanical context far reaching evolutionary principles that one would generally reserve for the living and even the mental realm.

Biological evolution has brought about smart self-organizing structures and systems, which entertain the functional integrity of living systems across many levels of organization from molecular systems to the macroscopic world. They orchestrate the flow of matter, energy, and information in a coherent way. It is more than reasonable to assume that these principles also apply to our perceptions, actions and cognitive processes. We have to come to terms with the idea that the dynamical objects which emerge on one organizational level become causally active on subsequent levels. Such a view is corroborated by the introductory models and introspective experiments in hearing that single out audible products of self-organization as parts of our conscious experience. They also demonstrate back action to the lower organizational levels and to the outside world. These dynamical entities pertain to the peripheral level of sensory processing, where measurements are relatively easy. Going up the information chain, neural synchronization phenomena provide the overarching model for large-scale integration and binding in brain dynamics [39]. Synchronization can be considered the dynamic glue that binds together information from different parts of the brain. The floating emergence and the interactions of more or less coherent neural ensembles make up the dynamical substrate of conscious perceptions. In the downward direction the same applies to the self-orchestration of actions. Collective neural activity coming from different parts of the brain acts downward and sets the boundary conditions for the concerted actions of biological nanomotors in effector organs that finally result in macroscopic actions.

It is the combination of bottom-up and top-down causation that allows complex behavior to emerge out of more simple components [42]. Across interlaced layers of emergent dynamical entities our mind obtains causal powers. Through our minds ideas and abstract concepts act back and impact on the physical and social world. The notion that our ideas, perceptions and actions are products of dynamical entities that mutually entertain their floating, seemingly immaterial existence is difficult to digest. We conceive of ourselves as autonomous beings endowed with a free will which controls our bodily actions in top-down fashion. The reverse direction is largely out of our focus. It can be highlighted by a thought-provoking experiment. This is the rubber hand illusion. It is based on perceiving synchronized visual and tactile information. A person watches the tickling of a rubber hand while the own visually hidden hand is synchronously tickled [43]. The integration of the synchronized visual and tactile signals in central processing causes a transfer of bodily ownership: More or less abruptly, the rubber hand is felt as one's own hand. Our brain and mind are intricately entangled with our body.

This refers back to the role of embodied cognition for promoting creative processes in science education, elaborated in a companion article [1]. Concrete experience helps in unfolding the potential structures and actions that a system affords. The integration into our intuitive repertoire assists modeling and knowledge progression. These few cursory examples are intended as epistemological and metacognitive incentives to reflect upon our status in the universe and to address the role of our mental and bodily activities in building models of the world. They complement the perspective on modeling and on technological applications of engineering self-organization from the preceding chapters and from the mechanics of self-organization and creative cognition described in paper [1]. The reflexive view on our creative minds is open-ended. It includes many loopholes and uncharted terrain. Beyond doubt this openness will keep the creative minds of present-day and future generations of STEM-learners motivated and busy.

7. Outlook: Creativity, STEM Excellence and Responsibility — A Delicate Balance

The present thematic and contextual approach to promote creativity and STEM excellence in intermediate and more advanced curricular settings is based on three main pillars.

- The methodological strand unfolds diverse facets of modeling. It includes functional models, dynamical models, and methods of model order reduction to decrease complexity. In spite of their simplicity the reduced models turn out to be highly powerful.
- The technological strand presents in a cursory way the orchestration of coherent quantum processes. They play an eminent role in the development of innovative technologies and in the creation of economic value. Motivating and empowering young creative minds to pursue careers as future professionals in these fields has become a major educational rationale.
- The conceptual strand unfolds self-organization as a cross cutting structuring principle that impacts on many domains of experience. The biological focus reflects back on our own role as intelligent beings and calls for the responsible use of the human creative powers to contribute to sustainable development.

The modeling part highlights the fertile role of analogies in promoting new knowledge. If differs somewhat from the ‘received’ view of analogies in education that considers analogy making a one-way process using knowledge from the base domain to elucidate the target domain [44]. The mapping is assumed to operate on sets of explicit roles similar to the formal steps in deductive and propositional reasoning [45]. This algorithmic view is somewhat distant from the actual practice of creating and using meaningful analogies. In teaching, the unidirectional view may hold for low complexity and surface analogies, but the students’ salient knowledge restricts the power of analogical reasoning on that level [46]. The actual advanced modeling processes deploy a more symmetric and productive reciprocity that depends on students’ expertise [47]. Analogies let the ideas flow back and forth between both domains. In this process, the ‘crystallized’ theoretical knowledge becomes fluid knowledge, ready for productive advancement. It is an indispensable part in the art of modeling to reflect structural and functional analogies with respect to their scope, viability and limitations. This meta-perspective finally illuminates why an analogy works successfully and how far it can be pushed forward.

With respect to educational practice the present approach calls for a much stronger coordination and a more coherent interlinking between the STEM subjects. However, in view of curricular traditions and historical priority claims of some fields, a successful implementation is challenging. An overarching focus on modeling as well as on the design of experiments and tools for problem solving has the potential to provide a unifying framework. It respects domain specific demands and reinforces methodological communalities. A general modeling competence in terms of understanding the nature of models and modeling processes is a desirable generic aim, but when it comes to actual quantitative modeling the devil is in the details. In that context the role of mathematics requires a critical consideration. At present mathematics education undergoes a transition phase towards strengthening real-world applications and mathematical modeling [48, 49]. Many curricula and educational standards address mathematical modeling of real-world problems, although the actual educational practice appears to lag behind considerably. Three main rationales in strengthening real-world contexts in mathematics curricula can be identified. They refer to mathematics as a tool for everyday life, to the real world as a vehicle for learning mathematics, and to engagement with the real-world as a motivation to learn mathematics [50]. These aims are primarily mathematics-centered. Their implementation can become problematic for several reasons. Using examples from physics or engineering as mere contexts and vehicles to convey mathematical formalism falls short of respecting the different domain specific embodiments of mathematical ideas and approaches, which are crucial for successful problem solving. Moreover, the somewhat hegemonistic connotation suppresses relevant contributions of these fields to mathematical modeling and the development of mathematical ideas. A recent empirical study on upper secondary school student's difficulties with mathematical modeling presents a sobering conclusion in this direction [51]. The students' problems and stumbling blocks did not reside in technical mathematical knowledge per se but "in the fundamentals of subjecting extra-mathematical situations to mathematical modeling". This is a challenge well known to physics educators: While students have no problem in dealing with linear relations of the standard form $y = mx + n$, well known from their mathematics lessons, they face considerable difficulties in applying this knowledge to interpret the proportionality between voltage and current, expressed by Ohm's relation $\mathrm{U} = \mathrm{RI}$. Therefore, we strongly advocate a clear orientation that focuses on qualitative modeling in terms of domain specific concepts before quantitative mathematical models can be deployed successfully. All this requires more efforts and efficient hybrid approaches of interlinking STE with M.

Along these lines the present sequence of exemplary experience-based encounters with self-organizing phenomena are arranged hierarchically along the continuum from qualitative to quantitative modeling. In secondary education the present qualitative approaches to self-organizing systems can be adapted to standard physics curricula on vibration and waves. Quantitative modeling of self-oscillation and regulation phenomena can be implemented successfully in upper secondary education. There are various graphical computer tools for modeling complex nonlinear dynamical systems in terms of stock and flow models, tailored for educational purposes [52]. However, as discussed above, before resorting to such tools, we strongly recommend starting from an algorithmic approach for numerically solving the dynamical equations, in order to develop an intuitive acquaintance for the operational meaning of state and rate variables and their interactions in the dynamical models. For university teaching, the present examples are intended as incentives to take up and integrate the idea of self-organization and emergence into existing approaches, in order to provide the basis for a better cross-domain exchange of ideas and expertise and for fostering the dialogue between different knowledge cultures.

The orientation towards the requirements of a competitive workforce underlies numerous educational reform policies. This shift of focus calls for a subtle rebalancing of economic versus humanitarian educational goals. Negotiating this dynamical equilibrium in our societies is a process with inherent conflicts as well as productive potentials. This should be reflected adequately in our efforts of promoting creativity and excellence. Throughout the history of human civilization creativity has been a major driving force. It can shape the world towards the better or the worse. Unleashing creativity and promoting excellence must therefore be accompanied by adequate educational measures that strengthen moral and social responsibility.

In view of the global impact of technological developments we are faced with the challenge to implement innovations in responsible ways. This includes creative ideas in conceiving new technologies to remedy detrimental effects of the foregoing technologies and to engineer a sustainable development of our societies and our natural habitat. Towards this aim, the ideas and the regulating principles of self-organization can provide a framework to guide our decisions. The global challenges to maintain and to improve the integrity of our natural environment call for coherent actions on the collective and the individual level.

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