Momentum flow in the electromagnetic field

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Newton's second law is equivalent to the continuity equation for momentum in integral form. This insight provides an alternative picture of forces in terms of momentum currents. In such a picture, the forces exerted on an object via a field are completely described locally in terms of the momentum current density of the field. The momentum current picture leads to a representation of Maxwell's stress tensor which is easy to visualize and to sketch quantitatively. Computer sketches of the momentum current distributions for a few simple examples are presented. In particular, it is shown that two like charges are "pulled apart" by their common electric field and that two parallel wires carrying electric currents in the same direction are "pushed together" by their common magnetic field.

I. INTRODUCTION

The interaction between two charged bodies, say, between one charged particle and another is typically described in the following way: The force acting on either particle is determined by the particle's own charge times the electric field which *would be* present at the particle's location if this particle itself were not charged. This way of speaking implies that each charged particle finds itself in the field of the other in spite of the fact that the actual field at every location is given by the superposition of the fields of both particles.

Obviously, this description requires one to keep two different electric fields in mind: one field for the calculation of forces and another field for the calculation of such things as the field energy or field momentum. This inconsistency is unsatisfying from a didactic point of view.

This typical description has yet other shortcomings. Usually when one body interacts with another via an intervening medium, for example, via a rope, rod or spring, a hydraulic fluid, a cushion of air,..., it is reasonable to ask about the path along which the force acts through the medium and, consequently, about the distribution of stress within this medium. But how about when this medium is an electric or magnetic field? When electric or magnetic interactions are treated in the fashion mentioned above, no straightforward answer can be given to this question. To see how misleading the typical description of the details about an electric interaction can be, consider the statement that two similarly charged particles "repel one another along their line of centers." This statement is incorrect if it is taken to mean that the force itself acts along this line since, after all, the electric field vanishes there at some point.

Of course, electromagnetic interactions can be described in a consistent and straightforward manner with the help of Maxwell's stress tensor. This kind of description is usually avoided, however, because of the mathematical complications thought to be associated with it. There does not seem to be a simple way to visualize the physics of the stress tensor. However, in this paper, it will be shown that such a simple way does, indeed, exist.

The local conservation of momentum and angular momentum leads to the identification of force as a momentum current[1,2] and torque as an angular momentum current.[3] Of course, this identification is not new: The concept of momentum current is familiar to the theory of fluids[4] and a discussion of angular momentum currents can be found, for example, in a popular European university text.[5] However, until now, there has not been a direct effort to apply the concept of momentum current more generally to other areas of physics.

In a recent paper,[6] momentum current distributions were investigated in static structures. This work is extended here to describe the distribution of momentum currents in the electromagnetic field. In particular, the momentum current picture provides a simple representation of Maxwell's stress tensor, i.e., of the momentum current density tensor. In this representation, the tensor field is easy to visualize and to sketch quantitatively.

II. MECHANICAL STRESSES AND THE MOMENTUM CURRENT DENSITY

The picture of a momentum current follows directly from the principle of the local conservation of momentum:

> The value of the momentum $\mathbf{p}$ contained within an arbitrary region R of space can change in time only if a net momentum current $\mathbf{I}$ flows through the (closed) boundary surface of R.

This statement implies a continuity equation for momentum:

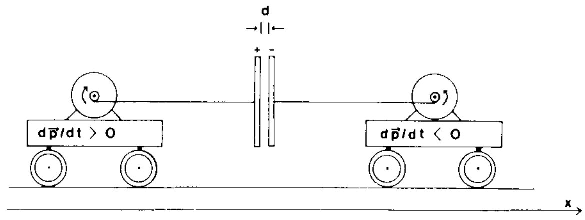

Fig. 1. Two wagons are connected across the plates of a capacitor by means of two ropes. The wagons are being pulled together with the help of motors attached to each wagon. For simplicity of argument, the motors are assumed to be winding up the ropes around spools in such a way that the separation between the capacitor plates remains constant throughout the process.

$$\frac{d\mathbf{p}}{dt} + \mathbf{I} = 0. \qquad (1)$$

A comparison of Newton's second law,

$$\frac{d\mathbf{p}}{dt} - \mathbf{F} = 0, \qquad (2)$$

with this continuity equation shows that "force" is identical to a negative momentum current:

$$\mathbf{F} = -\mathbf{I}. \qquad (3)$$

Figure 1 demonstrates that a momentum current is flowing in a static electric field. The momentum of the wagon on the left is increasing while the momentum of the wagon on the right is decreasing at the same rate. (We designate momentum to the right as positive.) The only connection between the wagons is the rope and the electrostatic field of the capacitor. Accordingly, (positive) momentum is flowing from the wagon on the right through the right piece of rope to the right-hand plate of the capacitor. From there, momentum continues flowing through the electric field to the left-hand plate, then through the left piece of rope and, finally, into the wagon on the left.

As is well known, a force $\mathbf{F}$ is related to a local quantity $\boldsymbol{\sigma}$, the mechanical stress tensor, according to

$$\mathbf{F} = \int d\,\mathbf{A}\cdot\boldsymbol{\sigma}. \qquad (4)$$

In a similar way, a momentum current $\mathbf{I}$ is related to a local quantity j, the momentum current density tensor, according to

$$\mathbf{I} = \int d\,\mathbf{A}\cdot\mathsf{j}. \qquad (5)$$

Thus, (3) with the help of (4) and (5) shows that the momentum current density is identical to the negative stress tensor[7,8]:

$$\mathsf{j} = -\boldsymbol{\sigma}. \qquad (6)$$

The reason for renaming (the negative of) the stress tensor as the momentum current density tensor lies in the advantage the latter name has in providing a picture of stress phenomena in terms of the flow of momentum in a way analagous to the picture of, say, electric phenomena in terms of the flow of electric charge. For example, equations analogous to (1) and (5) for the conservation of charge Q and for the charge current I_Q expressed in terms of the charge current density $\mathbf{j}_Q$ can be obtained by replacing $\mathbf{p}$ and $\mathbf{I}$ in (1) with Q and I_Q and by replacing $\mathbf{I}$ and j in (5) with I_Q and $\mathbf{j}_Q$, respectively. Of course, Eqs. (1) and (5) are more complicated than their electric analogs: since Q is a scalar,

I_Q must also be a scalar and $\mathbf{j}_Q$ a vector; however, since **p** is a vector, **I** must also be a vector and j must be a second rank tensor. Nevertheless, it is just as easy to visualize a momentum current as it is to visualize a charge current if (1) and (5) are considered component by component. Accordingly, (1) is equivalent to three "scalar" continuity equations

$$\frac{dp_x}{dt} + I_x = 0,$$
$$\frac{dp_y}{dt} + I_y = 0,$$
$$\frac{dp_z}{dt} + I_z = 0. \tag{7}$$

Here I_x, I_y, and I_z are "scalar" currents for the flow of the x, y, and z components of momentum, respectively (for short, the x momentum, y momentum, and z momentum). Similarly, from (5) we have

$$I_x = \int d\,\mathbf{A}\cdot\mathbf{j}_x,$$
$$I_y = \int d\,\mathbf{A}\cdot\mathbf{j}_y,$$
$$I_z = \int d\,\mathbf{A}\cdot\mathbf{j}_z, \tag{8}$$

where $\mathbf{j}_x$, $\mathbf{j}_y$, and $\mathbf{j}_z$ are "vector" current densities for the flow of x, y, and z momentum, respectively. Mathematically speaking, $\mathbf{j}_x$, $\mathbf{j}_y$, and $\mathbf{j}_z$ are simply the projections of j along the x, y, and z directions, respectively, and, in a matrix representation of j, make up the columns (or rows) of the (symmetric) j matrix.

The advantage of dealing with the three independent "vector" fields $\mathbf{j}_x$, $\mathbf{j}_y$, and $\mathbf{j}_z$ is that these can be pictorially represented in the usual manner in terms of stream lines. The stream lines of, say, $\mathbf{j}_x$ show how x momentum flows throughout the physical system being considered.

For the complete description of the flow of momentum in a system, three stream-line pictures are required. If, however, the system is sufficiently symmetric, three or two of these pictures become identical by a suitable orientation of coordinates so that one need sketch only one or two such pictures.

III. MAXWELL'S STRESS TENSOR

Already in the last century, Faraday recognized that mechanical stresses are present within the electromagnetic field: tension parallel to the **E**- and **H**-field lines and pressure perpendicular to them. Maxwell was the first to express these stresses mathematically in terms of the electric and magnetic field vectors **E** and **H**. The resulting expression for the stress tensor of the electromagnetic field came to be known as Maxwell's stress tensor. The negative Maxwell's stress tensor represents the momentum current density tensor of the electromagnetic field

$$j_{kl} = \tfrac{1}{2}\delta_{kl}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) - E_k D_l - B_k H_l. \tag{9}$$

Here **D** is the electric displacement vector, **B** is the magnetic induction, and δ_{kl} is the Kronecker delta. From (9), it can be seen that the momentum flow is not affected by reversing the direction of the fields.

Our tool for calculating the momentum current distribution in the electromagnetic field is completely contained in Maxwell's stress tensor. For this reason, it is helpful to consider some simple rules for drawing the momentum current density stream lines representing (9).

The tensor (9) becomes simpler in the special case that (i) one of the two fields **E** or **H** is zero and (ii) the medium is isotropic, i.e., the dielectric constant ϵ or the magnetic permeability μ is a scalar:

$$\mathrm{j} = \left(\frac{\epsilon}{2}\right)\begin{pmatrix} E^2/2 - E_x^2 & -E_xE_y & -E_xE_z \\ -E_yE_x & E^2/2 - E_y^2 & -E_yE_z \\ -E_zE_x & -E_zE_y & E^2/2 - E_z^2 \end{pmatrix}. \tag{10}$$

The columns [or rows, since (j_{kl}) is symmetric] of the stress tensor (10) represent the current density "vectors" for the x, y, and z momentum, respectively:

$$\mathbf{j}_x = (E^2/2 - E_x^2, -E_xE_y, -E_xE_z),$$
$$\mathbf{j}_y = (-E_yE_x, E^2/2 - E_y^2, -E_yE_z),$$
$$\mathbf{j}_z = (-E_zE_x, -E_zE_y, E^2/2 - E_z^2). \tag{11}$$

We now consider one of the three current density "vectors," say, $\mathbf{j}_x$, for various orientations of the vector **E**.

If $\mathbf{E} = (\pm E, 0, 0)$, then $\mathbf{j}_x = (-E^2/^2, 0, 0)$ and we formulate Rule I.

Rule I: Everywhere where **E** is parallel or antiparallel to the x axis the x-momentum current density $\mathbf{j}_x$ points in the negative x direction [Fig. 2(a)].

This is in agreement with the example illustrated in Fig. 1. In view of the rule given in Ref. 6, this expresses the local Faraday tension along **E**.

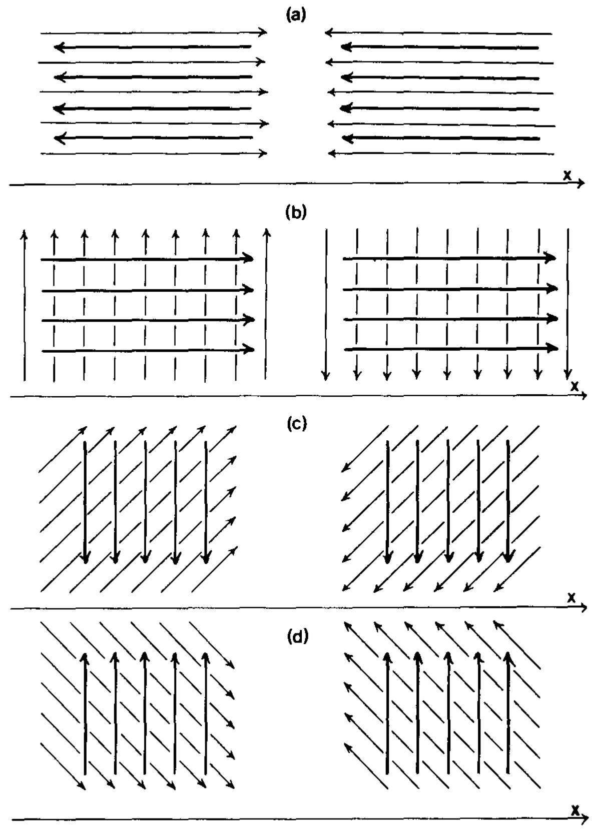

Fig. 2. Illustration of some rules useful for sketching the direction of the $\mathbf{j}_x$ stream lines (heavy lines) for various orientations of **E** (light lines): (a) **E** parallel or antiparallel to the $+x$ direction. (b) **E** parallel or antiparallel to the $+y$ direction. (c) **E** makes an angle of $+\pi/4$ or $-3\pi/4$ with the $+x$ direction. (d) **E** makes an angle of $-\pi/4$ or $+3\pi/4$ with the $+x$ direction.

If $\mathbf{E} = (0, \pm E, 0)$ or $E = (0, 0, \pm E)$, then $\mathbf{j}_x = (+E^2/2, 0, 0)$ and we formulate Rule II.

Rule II: Everywhere where $\mathbf{E}$ is perpendicular to the x axis, the x-momentum current density $\mathbf{j}_x$ points in the positive x direction [Fig. 2(b)].

This expresses the local Faraday pressure perpendicular to $\mathbf{E}$.

Finally, if $\mathbf{E} = (+E/\sqrt{2}, \mp E/\sqrt{2}, 0)$ or $\mathbf{E} = (-E/\sqrt{2}, \pm E/\sqrt{2}, 0)$, then $\mathbf{j}_x = (0, +E^2/2, 0)$ or $\mathbf{j}_x = (0, -E^2/2, 0)$, respectively, and we formulate Rule III.

Rule III: Everywhere where $\mathbf{E}$ (taken to lie in the xy plane) makes an angle of $\mp \pi/4$ or $\pm 3\pi/4$ with the x axis, the x-momentum current density $\mathbf{j}_x$ points in the $\pm y$ direction [Fig. 2(c) and (d)].

Analogous rules also pertain for the y- and z-momentum current densities.

Finally, the directions $\mathbf{j}_x, \mathbf{j}_y$, and $\mathbf{j}_z$ are related by Rule IV.

Rule IV: The x-, y-, and z-momentum current density "vectors" are perpendicular to one another at every point of the field.

These rules[9] remain literally valid if $\mathbf{E}$ is everywhere replaced with $\mathbf{H}$. They are, of course, restricted to isotropic media, i.e., media for which $\mathbf{E} \| \mathbf{D}$ and $\mathbf{H} \| \mathbf{B}$. Obviously, the field stresses can be "picked apart" in this way (rules I–IV) for any arbitrary, pure electric or magnetic field.

These rules might, at first sight, appear rather counterintuitive. For example, one might wonder why the momentum current does not reverse direction when the field is reversed. To better understand this,[10] recall Fig. 1 and notice that the polarity of the capacitor does not matter: the carts would still be pulled together if the polarity were reversed. In order that the momentum current flowing between the carts be reversed, the ropes would have to be replaced with rods *pushing* on the capacitor. But then the plates would have to carry *like* charges. In this case, the field between the plates would be predominantly *perpendicular* to the x direction [Fig. 2(b)], which is just the condition (Rule II) for the x-momentum current to flow in the other direction.

We recommend the reader to direct his attention to these rules while considering the computer sketches of momentum current distributions discussed in the next section.

IV. MOMENTUM CURRENTS IN SOME SIMPLE FIELDS

We first consider the momentum current density in a plane perpendicular to the axis of a uniform line charge. Considering line charges has the advantage that the resulting stream line pictures are identical to those for current carrying wires except for an overall reversal of the sense of direction (see below). The magnitude of the momentum current density drops off as $1/r^2$, where r is the separation between the axis of the line charge and the considered point. Notice that the x-, y-, or z-momentum current density stream lines do not go off to infinity (as opposed to the field lines of the electric field) but, rather, they return back to the line charge. Calculations show that the stream lines close in exact circles. The momentum current density in the Coulomb field of a point charge has the same form, only the density of lines drops off more rapidly, i.e., as $1/r^4$ instead of $1/r^2$. Figure 3(a) and (b) shows sketches of the $\mathbf{j}_x$ and $\mathbf{j}_y$ stream lines, respectively, for the field of a line

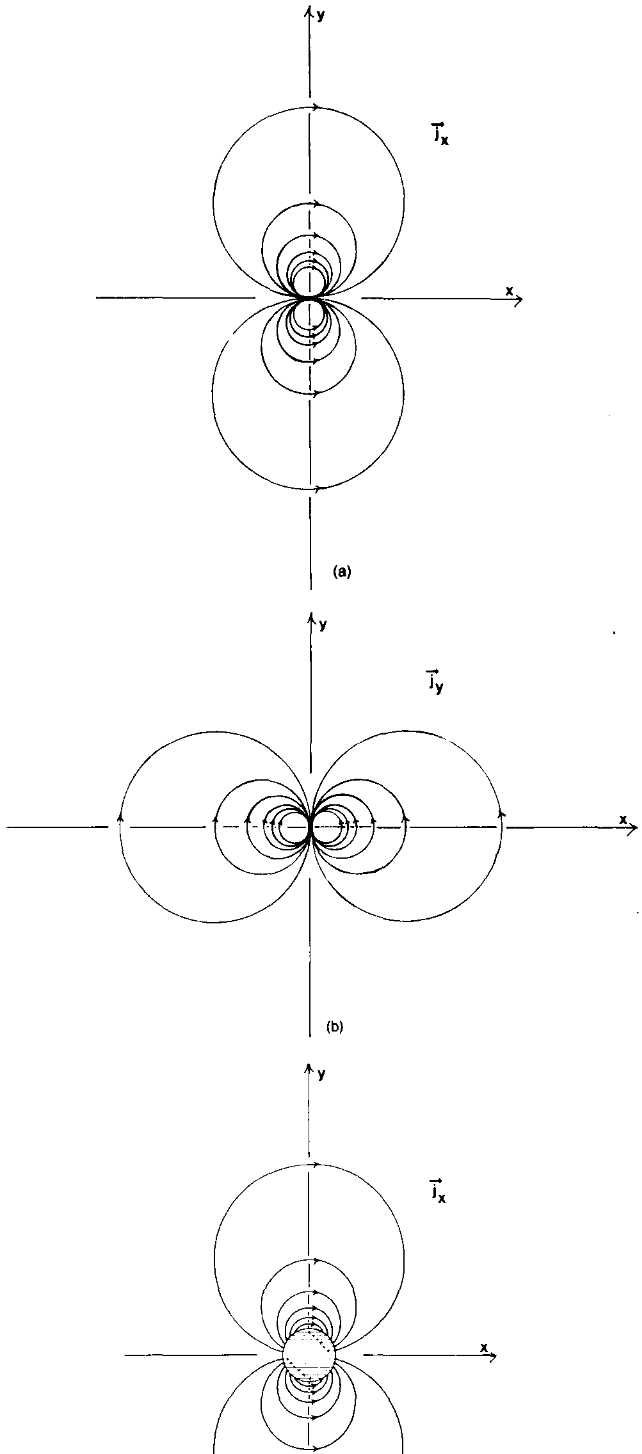

Fig. 3. (a) Sketch of some $\mathbf{j}_x$ stream lines in a plane perpendicular to the axis of a uniform line charge. The circles become denser in the vicinity of the line charge. The innermost circles are not represented. If the sense of direction of the stream lines is reversed, this becomes the sketch of the $\mathbf{j}_x$ stream lines in a plane perpendicular to the axis of a line current. (b) Sketch of some $\mathbf{j}_y$ stream lines in the field of line charge. (c) Sketch of some $\mathbf{j}_x$ stream lines in the field of a cylindrical object with uniform surface charge. The stream lines are shown to loop through the body of the object itself.

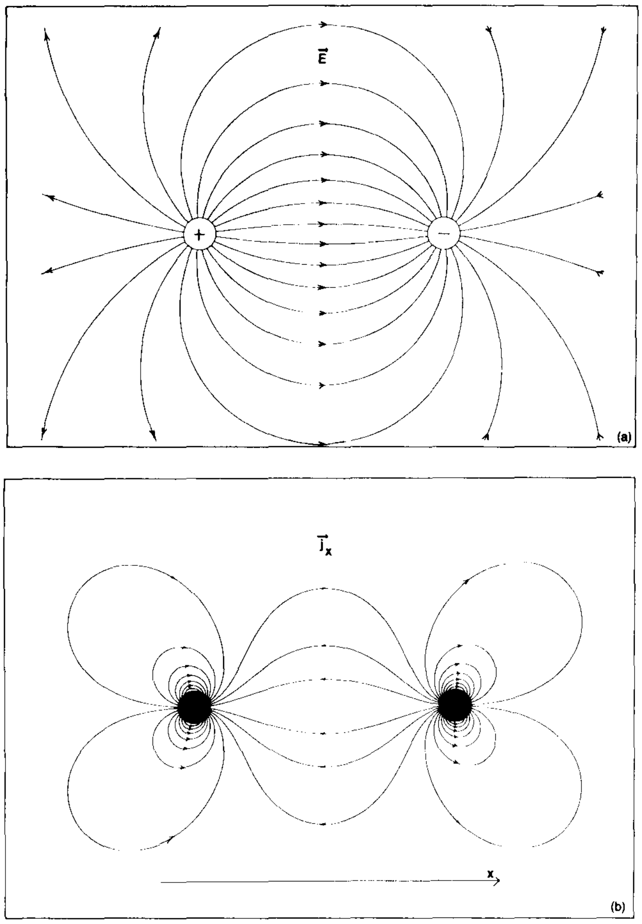

Fig. 4. (a) Sketch of the electric field lines in a plane perpendicular to the (parallel) axes of two equally but oppositely charged cylinders. (b) Sketch of the j_x stream lines in the electrostatic field of Fig. 4(a). If the sense of direction of the stream lines is reversed, this becomes the sketch of the j_x stream lines in the magnetic field of two repelling line currents.

charge. Notice that these stream lines are not cylindrically symmetric like the electric field itself. This is because a particular direction in space was selected out in each case by the corresponding choice of coordinate axes. From Fig. 3(a) and (b), it is obvious that the divergence of the momentum current density field vanishes, i.e., as expected, there are neither sources nor sinks for x or y momentum (or z momentum) in this electric field.

If the line charge is replaced by a cylinder with a uniform surface charge, the electric field and, accordingly, the momentum current distribution outside the cylinder remains unchanged. However, the stream lines of the x-, y-, and z-momentum current density must now close within the matter of the charged object [Fig. 3(c)]. The stream lines transfer from field to matter at the location of the surface charges. This means the object itself will be under tension. The existence of this tension can be observed by an increase in radius of such an object, for example, by the swelling of a balloon upon charging its surface.

Figure 4(a) shows the electrostatic field lines sketched in a plane perpendicular to the (parallel) axes of two equally but oppositely charged cylinders. If not somehow held apart, the cylinders will approach one another along the line connecting their axes. If we orient the x axis along this line, this means that x momentum flows from the cylinder at the right through the electrostatic field to the cylinder at the left.

With the help of the momentum current density (Maxwell's stress) tensor, it is possible to calculate how x momentum flows through the electrostatic field sketched in Fig. 4(a). The result is shown in Fig. 4(b). The interesting thing about this x-momentum current density distribution is that it directly gives the force acting on the charged objects. The force acting on either object is simply proportional to the number of stream lines which penetrate its surface. It is not possible to read this force so directly from a sketch of the electrostatic field itself. Two types of x-momentum current density stream lines can be recognized in Fig. 4(b):

(i) Stream lines which return to the object of their origin and end there.The density of returning stream lines is a measure of the internal (material) stress in the corresponding charged object. Such an internal stress manifests itself in a deformation of the object. Recalling from above that there is tension everywhere where the $\mathbf{j}_x$ stream lines point in the negative x direction, we see here that there is tension over the entire surface of the object. In other words, the electric field "tugs" everywhere at the surface of the object as if to pull it apart.

(ii) Stream lines which start from one object and end at the other. The force with which both objects attract one another is given alone by those stream lines which connect the objects together, since this force is nothing more or less than the net flow of momentum from one object to the other. Notice that these lines are not constrained to the line of centers connecting the objects together: The force between two oppositely charged objects does not act entirely along their line of centers. Accordingly, it is misleading to say that "the force between two opposite charges acts along their line of centers."

Figure 5(a) shows the electrostatic field sketched in a plane perpendicular to the (parallel) axes of two equally and similarly charged cylinders. Notice that the electric field vanishes in the middle of the line connecting their axes. But momentum cannot be transported through the electric field at locations where the field itself vanishes. Thus, no momentum can flow from one cylinder to the other along a line connecting their axes. For the same reason, no momentum can flow from one spherically charged object or point charge to another along their line of centers. Accordingly, it is misleading to say that "the force between two similar charges acts along their line of centers." An exact calculation with the help of the stress tensor confirms this expectation. Figure 5(b) shows the x-momentum current density stream lines for the electric field of Fig. 5(a): No stream line at all can coincide with the line of centers.

At the midplane between the objects, the stream lines point in the positive x direction, i.e., x momentum flows from the object at the left to the object at the right. Nevertheless, these left-to-right stream lines point in the negative x direction immediately at the surface of both objects. This means that the field tugs at the surface of both objects: The objects are pulled apart by their common electrostatic field.

Taken together, these two examples lead to the following insight into the nature of electric attraction and repulsion: The tension along the electric field lines at the surfaces of two similarly charged objects is responsible for their repulsion as well as the tension along the electric field lines at the surfaces of two oppositely charged objects is responsible for their attraction.

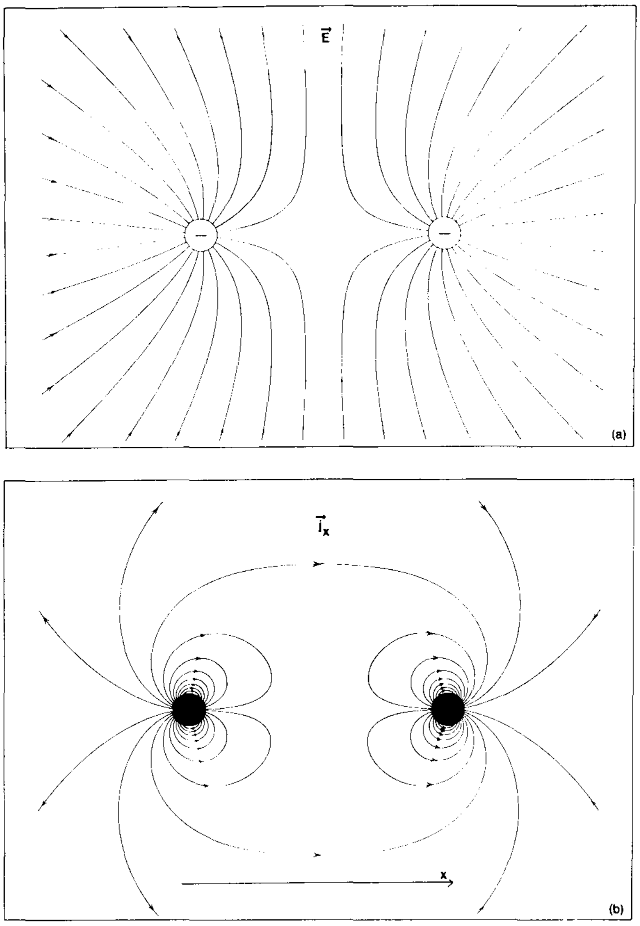

Fig. 5. (a) Sketch of the electric field lines in a plane perpendicular to the (parallel) axes of two equally and similarly charged cylinders. (b) Sketch of the $\mathbf{j}_x$ stream lines in the electrostatic field of Fig. 5(a). If the sense of direction of the stream lines is reversed, this becomes the sketch of the $\mathbf{j}_x$ stream lines in the magnetic field of two attracting line currents.

It is not difficult to show that the $\mathbf{j}_x$ stream lines in the magnetic field of two repelling (attracting) line currents are identical to those in the electric field of two attracting (repelling) line charges except for an overall reversal of the sense of direction. Thus, by drawing analogous pictures for the momentum currents in the magnetic fields of current carrying wires, it can be seen that parallel wires in which the electric currents have opposite directions are "pushed apart" by their common field, whereas wires with electric currents of the same direction are "pushed together" by their common magnetostatic field.

These considerations involve more than just a manner of speaking. For example, two similarly charged elastic bodies—say, two similarly charged soap bubbles or gas clouds—would display quite different deformations in the vicinity of one another depending upon whether they are pushed or pulled apart by their common electric field. Thus, the insight provided by the considerations of computer sketches similar to those shown in Figs. 3(c), 4(b), and 5(b) (but including the combined effects of both electric and magnetic stresses) can be expected to be of practical value for real, physical problems involving interactions between deformable charge configurations, for example, as one might find in the plasma of a fusion or MHD machine.

V. CONCLUDING REMARKS

This paper has shown how stresses within the electromagnetic field can be understood on the basis of the momentum current picture. An important conceptual aspect of this picture is that it provides a local-causes description of mechanics. An important practical aspect is twofold: (1) diagrams of the momentum current density stream lines enable one to easily visualize the stress state of the field; (2) these diagrams enable one to easily read the value of the net momentum current flowing into the bodies (= force acting on the bodies) where they coincide with the field sources. Accordingly, the stress state of any such body can easily be inferred.

[1]F. Herrmann, "Mechanik—Abriss einer Neudarstellung," in *Konzepte eines zeitgemässen Physikunterrichts*, edited by G. Falk and F. Herrmann (Schroedel-Verlag, Hannover, 1979), Vol. 3, p. 80; also, Vol. 5 (1982).

[2]A. A. diSessa, Am. J. Phys. **48**, 365 (1980).

[3]F. Herrmann and G. B. Schmid, "Rotational dynamics and the flow of angular momentum" (in-house report).

[4]L. D. Landau and E. M. Lifschitz, *Fluid Mechanics* (Pergamon, Oxford, 1959), Chap. I, Sec. 7.

[5]G. Falk and W. Ruppel, *Energie und Entropie* (Springer-Verlag, New York, 1976), pp. 47–51.

[6]F. Herrmann and G. B. Schmid, Am. J. Phys. **52**, 146 (1984).

[7]L. D. Landau and E. M. Lifschitz, *Theory of Elasticity* (Pergamon, Oxford, 1959), Chap. I, Sec. 2.

[8]L. D. Landau and E. M. Lifschitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960), Chap. II, Sec. 15.

[9]The traditional way of looking at these rules is to consider a system in electrostatic equilibrium and to imagine it to be divided into two parts by an arbitrary closed surface S (which may lie partly or even entirely in matter-free space). Then it is not difficult to show that the local electric field vector bisects the angle between the normal to an arbitrary element of surface $d\mathrm{S}$ of S and the surface force $\mathbf{F} = (-\mathbf{j}_x \cdot d\mathrm{S}, -\mathbf{j}_y \cdot d\mathrm{S}, -\mathbf{j}_z \cdot d\mathrm{S})$ at $d\mathrm{S}$. By rotating the surface element $d\mathrm{S}$ so that it makes various angles with the direction of the field, conclusions about the field stresses similar to our Rules I–III of Sec. III are obtained. See, for example, M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism* (Blackie, London and Glasgow, 1947), p. 107.

[10]We thank one of our referees for suggesting this example.