# How does the ball-chain work?

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A well-known collision experiment can be carried out with an arrangement of several elastic balls suspended in a horizontal row. As we have shown in a previous article a necessary condition for the observed, simple behavior of this arrangement during collision is that the perturbation propagates throughout the system without dispersion. In the present paper, we show that the arrangement can be described by a series of spatially separated masspoints and springs of a special type: the exponent of the force law of the springs is 1.5 according to a theory of H. Hertz. It follows that the first collision sequence of such an experiment is not completely dispersion free. Indeed, slight dispersion during the first collision sequence creates the conditions for the total absence of dispersion in all the subsequent collisions.

## I. INTRODUCTION

Figure 1 shows a well-known arrangement of elastic balls commonly used to demonstrate the law of conservation of momentum. Each ball is suspended by two threads with neighboring balls touching one another. If a certain number of balls is displaced from its position of equilibrium and then allowed to swing against the other balls at rest the same number of balls moves away to the opposite side of the chain after the collision as had initially been displaced.

The observed behavior is in agreement with the laws of conservation of energy and momentum. Nevertheless, these two conservation laws are not in themselves sufficient to explain this behavior whenever the chain consists of more than two balls. It has been shown by Herrmann and Schmälzle[1] that, for sufficiency, an additional condition is required, namely, the absence of dispersion. The simple, familiar behavior of the chain mentioned above follows only if the perturbation created by the collision propagates through the chain without changing shape. This conclusion is independent of the particular reason why dispersion might be absent.

A trivial case of a dispersion-free arrangement is obtained when the balls are suspended with a small interval between neighboring balls. In the case of a single incident ball, the resulting propagation is a succession of independent single collisions between neighboring balls. In this case, the momentum of the incoming ball is completely transferred to the second ball before the second ball is in contact with the third one and so on. The same behavior is observed in an air-track experiment, whereby the balls are replaced with gliders: each glider is in turn equipped with a spring bumper and all the gliders are arranged in such a way that there is an interval between neighboring gliders. If one glider is moving toward the other gliders (at rest), it can easily be seen that the incoming glider decelerates until it is completely at rest while the second one accelerates from rest to the incident velocity. Thus the total momentum of the incoming glider is transferred to the next one and so on.

However, how can the absence of dispersion in the familiar chain of balls be explained when each ball is initially in physical contact with its neighbors? It is easy to see that this question is not a trivial one if the corresponding air-track experiment is carried out. The motion of the individual gliders immediately after the collision seems to be in complete disorder. Apparently, the glider arrangement is not a good model of the ball chain. What is the fundamental difference between the ball chain and the glider chain? Could it be that the inertial and the elastical components of the system cannot be separately modeled? Or does the glider model contain other inadmissible simplifications? The answer to these questions is equivalent to finding the reason for the absence of dispersion within the chain of balls.

## II. COLLISION BETWEEN TWO BALLS

Let us first consider the simplest possible collision experiment: a central elastic collision between one incident ball and an identical ball at rest. In this case dispersion is principally absent and the conservation laws of energy and momentum completely determine the outcome of the experiment. For this reason, the glider model experiment behaves in the same way as the actual ball experiment: the incident body is at rest after the collision and the target body moves away with the incident velocity. A theory of elastic two-body collisions with deformation was developed in 1881 by H. Hertz.[2] Hertz calculated the deformation of two elastic bodies with convex surfaces as a function of their being pressed one against the other. He then applied his results to the two-body collision problem. In particular, he computed the collision time, i.e., the interval between the instant of first contact and the instant of separation of the balls, as a function of the velocity of the incoming ball. The following results of Hertz's work are of importance for our analysis:

—The stress is much greater in the immediate neighborhood of the region of contact than throughout the rest of

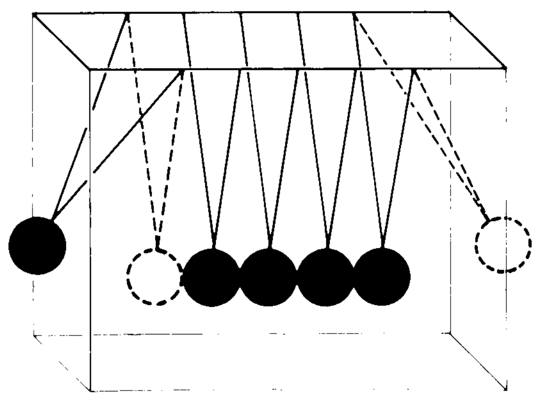

Fig. 1. Arrangement of elastic balls commonly used to demonstrate the law of conservation of momentum.



 

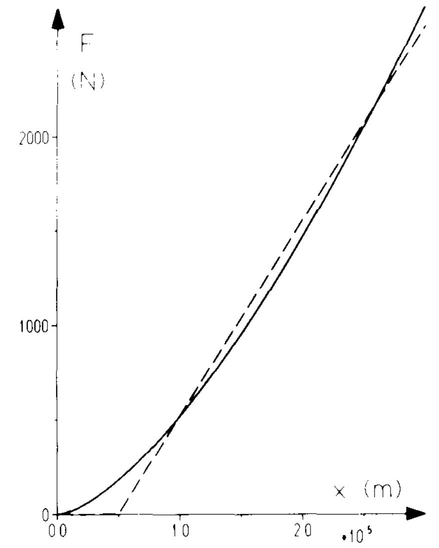


Fig. 2. Force $F$ as a function of the deformation $x$ in one dimension for a steel ball, 5 cm in diameter, according to the theory of Hertz (solid line). The dashed line represents an approximation of this curve by two straight lines.

the ball. Thus the elastic and inertial properties of the collision can be considered to be spatially separated. The material in the neighborhood of the contact region acts as a spring. In this region, energy can be stored by deforming the material. On the other hand, the bulk of the ball acts as an undeformable massive body in which energy can be stored only by storing momentum.

—The principal difference between the balls and the gliders is the manner in which the "springs" behave: The springs of the glider model are normal "Hookian" springs: the force $F$ is

$$F = kx. \qquad (1)$$

Here, $x$ is the displacement of the elastic system from its equilibrium position and $k$ is the spring constant.

The "springs" of the touching balls obey another law (see Fig. 2):

$$F = k' x^{3/2}. \qquad (2)$$

In the following, we will call this relation "Hertz's law."

The slope of the relations (1) and (2) is particularly interesting for us. For Hooke's law (1) the slope has the same value for all $x$, even for $x = 0$. It represents the "stiffness" of the spring. For Hertz's law (2), the slope is zero for $x = 0$ and increases with $x$. This might be considered as a law for a spring which becomes harder and harder as it is compressed. Hertz's law could be approximately represented by two straight lines as shown in Fig. 2. This, however, is the exact shape of the force law for the nontouching air-track gliders: The horizontal portion of the curve corresponds to the interval between the gliders, the linearly rising portion corresponds to Hooke's law for the spring bumpers. We conclude from this that the dispersion in a system with Hertzian springs is weaker than that in a system with Hookian springs.

Is this reduction of the dispersion sufficient to explain the outcome of the ball collision experiment? This question can be answered only after a quantitative consideration of the situation. Therefore, we simulated various collision experiments on a computer in a way to be detailed in Sec. III.

The computer simulation has the advantage that parameters of the chain can be arbitrarily varied and the effect of these variations can be studied.

## III. COMPUTER SIMULATION OF COLLISION EXPERIMENTS

During a collision process, the momentum spreads throughout the entire target ball. For objects as small as the balls in our collision experiment, this occurs over a much shorter time interval than the actual momentum transfer process itself. One might say that the "spring" through which the momentum flows from one ball into the other is a bottleneck for the momentum transfer. The collision time is large with respect to the period associated with the lowest (nontranslatory) eigenfrequency of the ball, i.e., with respect to the transit time across the ball for elastic waves. The only eigenfrequency which will be effectively excited by the collision is the "zeroth eigenfrequency," i.e., the translation. Consequently, the balls of the ball chain can be treated as mass points.

In view of these remarks, we represent a chain of $n$ interacting balls by $n$ mass points of mass $m$ each and $n - 1$ springs. Each spring satisfies the following general relation between the force and the distortion of the spring from its equilibrium length:

$$F = k(x_{i-1} - x_i)^r. \qquad (3)$$

Here $x_i$ is the displacement of the $i$th masspoint from its position of equilibrium. This mass–spring system is represented by a set of $n$-coupled differential equations:

$$\begin{aligned} &m\ddot{x}_1 + k(x_1 - x_2)^r = 0,\\ &m\ddot{x}_2 - k(x_1 - x_2)^r + k(x_2 - x_3)^r = 0,\\ &\quad\vdots\\ &m\ddot{x}_i - k(x_{i-1} - x_i)^r + k(x_i - x_{i+1})^r = 0,\\ &\quad\vdots\\ &m\ddot{x}_n - k(x_{n-1} - x_n)^r = 0. \end{aligned} \qquad (4)$$

The exponent $r$ is varied from one computer experiment to the next. For Hookian springs $r$ equals 1.0 and for Hertzian springs, 1.5. However, in the simulation other values of $r$ have also been chosen. We take $m = 0.512$ kg and $k = 1.638 \times 10^{10}$ N/m$^{3/2}$ to be in agreement with the experiments to be described in Sec. IV. $k$ is calculated according to[3]

$$k = (2/3)[E/(1 - \sigma^2)]\,[(1/2)R\,]^{1/2},$$

where $R$ is the radius of the balls, $\sigma$ the Poisson ratio, and $E$ Young's modulus for the material of the balls. The set of equations (4) doesn't take into account that the interaction between neighboring bodies is switched off whenever $x_{i-1} - x_i < 0$. This condition however has been considered in the computer program.

The computer calculated and plotted as a function of time:

—the displacement of every ball from its equilibrium position;

—the momentum of every ball;

—the force exerted by one ball on the next one.

The results reproduced in Figs. 3–5 are obtained by simulating chains with one incoming body colliding with four bodies at rest. The velocity of the incoming body immediately before the impact was choosen to be 0.443 m/s. This corresponds to an initial vertical displacement of the incident ball by 1 cm from its equilibrium level.

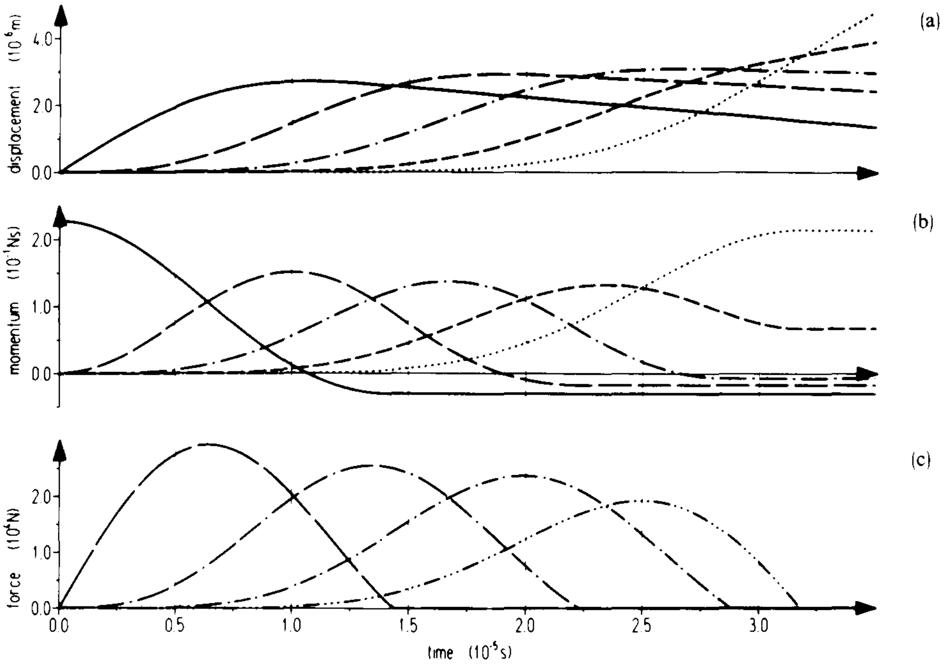


Fig. 3. Computer simulation of a collision experiment for one incoming ball colliding with 4 balls at rest. The exponent $r$ in equation (3) is 1.0. (a) The displacement from equilibrium of each ball (upper plot) as a function of time. The first ball is the incoming ball. Legend: ———— (1st ball); — — — — (2nd ball); —·—·— (3rd ball); - - - - - - - (4th ball); .................. (5th ball). (b) The momentum of each ball (middle plot) as a function of time. (c) The force of one ball exerted on the next as a function of time. Legend: — — (1st ball to 2nd); - —- — (2nd ball to 3rd); —·—·— (3rd ball to 4th); —·····—··· (4th ball to 5th).

Figure 3 shows the results for $r = 1.0$, i.e., Hooke's law. As could be expected, dispersion is important. This can be seen in all of the three plots of the figure. The upper two plots demonstrate that, after the collision process, all bodies are moving: The first three bodies move backward, the fourth and the last one move forward. After the collision, the fourth body has a momentum which represents about 29% of the momentum of the incoming body before the initial impact (and not 0% as might be expected). The lower plot of Fig. 3 gives a good idea of the dispersion of the energy–momentum transport throughout the chain. According to Newton's second law, the force is equal to the rate of change of momentum. Thus the lower plot of Fig. 3 tells us, that the exchange of momentum between the first and the second body occurs over a smaller time interval than the momentum exchange between each following pair of bodies.

Figure 4 represents the corresponding curves for mass-points which are coupled by Hertzian springs. Here, clearly, the dispersion is weaker. After the collision, the fourth

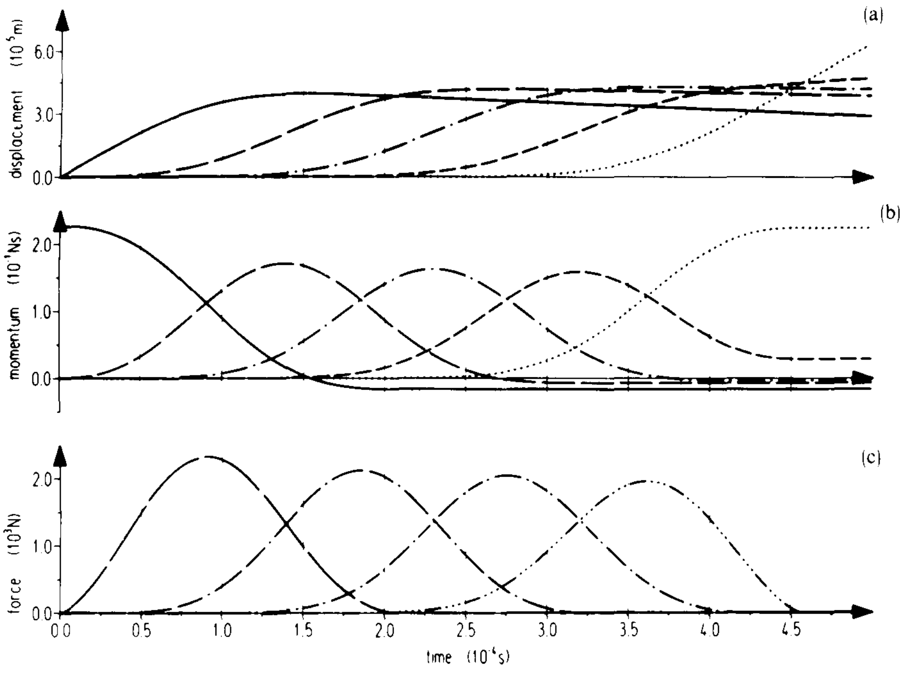


Fig. 4. Same as Fig. 3, except with $r = 1.5$.

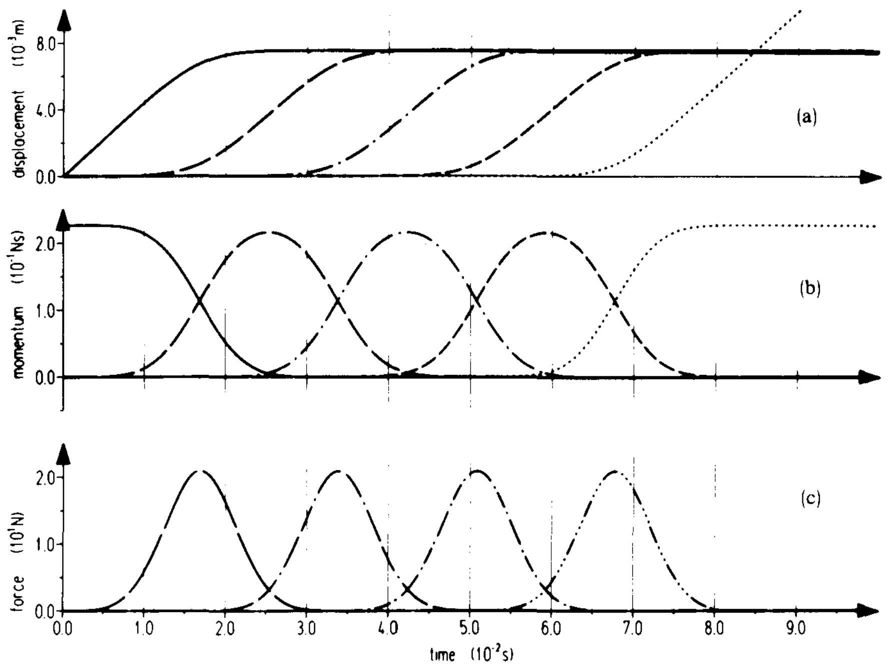


Fig. 5. Same as Fig. 3, except with $r = 4.0$.

body has no more than 12% of the initial momentum of the incident body. The curves in the lower part of Fig. 4 are more similar to one another than those in Fig. 3. Nevertheless, one would not call the dispersion in this experiment negligible.

We therefore tried force versus displacement laws with exponents $r$ higher than 1.5 and found that for $r \geqslant 4$ dispersion is almost completely absent. Figure 5 shows the diagram corresponding to $r = 4.0$.

What is to be concluded from these results? Is Hertz's law not in agreement with the experiment? Do real balls behave according to a law with $r > 1.5$? To infer the actual exponent, we carried out a real experiment.

## IV. EXPERIMENTAL DETERMINATION OF THE EXPONENT IN THE FORCE VERSUS DISPLACEMENT LAW

Chains of different lengths consisting of from 2 to 15 hardened steel balls, each 5 cm in diameter, were set up. Each ball was suspended by two threads and in contact with its immediate neighbors. A convenient variable for the comparison of experiment with theory is the propagation time for a perturbation through the ball chain. We define the propagation time as the interval between the instant of first contact of the first and the second ball ($t = 0$ in the computer diagrams) and the instant of separation of the last two balls from each other (the instant of intersection between the position versus time curves of the last two balls). The propagation time was measured in the following way: an electronic clock was triggered by the electrical contact established when the first ball touched the second one and it was stopped when the electrical contact between the two last balls was interrupted. Table I shows the propagation time for chains of 2,5,10, and 15 balls (column 2), respectively. In the case of the two-ball chain, the same contact serves to both start and stop the clock. The "propagation time" of 0.171 ms in this case is in good agreement with the collision time calculated according to Hertz's theory: 0.177 ms.

From the length of the chain (column 3) and the propagation time, a propagation speed can be calculated (column 4). It should be noted that this propagation speed is not, as might be supposed, the speed of sound in steel (which is 5100 m/s). Indeed, a perturbation propagates within the chain with a speed which is an order of magnitude smaller than the speed of sound and which varies as a function of the momentum of the incoming ball. This can be understood by recalling that, for the chain of balls, there is an inherent time delay associated with the compression and relaxation of each ball during the collision. This is in addition to the propagation time associated simply with the velocity of sound within the medium as, say, within a solid steel rod. In our example, this additional time delay ($\approx 0.17$ ms for the collision between two steel balls) is approximately 10 times greater than the propagation time for sound ($\approx 0.01$ ms within either ball). This backs up the qualitative discussion in Sec. II. In analogy to an electrical delay line consisting of an arrangement of capacitors and inductances, the ball chain might be called a mechanical delay line.

As the computer simulation shows, the propagation time depends strongly on the value of the exponent $r$. Thus by measuring the propagation time, we have a sensitive criterion upon which to base an inference about the value of $r$. Table II gives the propagation time read off Figs. 3–5 plus

Table I. Propagation time and speed for a perturbation through chains consisting of 2, 5, 10, and 15 balls.

| Number of balls | Propagation time ($10^{-3}$ s) | Length of the chain (m) | Propagation speed (m/s) |
|---|---|---|---|
| 2 | $0.171 \pm 0.003$ | 0.10 | 585 |
| 5 | $0.434 \pm 0.003$ | 0.25 | 576 |
| 10 | $0.845 \pm 0.002$ | 0.50 | 592 |
| 15 | $1.278 \pm 0.004$ | 0.75 | 587 |

Table II. Propagation times for a chain of 5 balls from computer simulation with several exponents $r$.

| $r$ | Propagation time (s) |
|---|---|
| 1.0 | $3.2\times10^{-5}$ |
| 1.5 | $4.5\times10^{-4}$ |
| 2.0 | $2.7\times10^{-3}$ |
| 4.0 | $8.5\times10^{-2}$ |

one additional figure (not shown) for an exponent of $r = 2.0$. Comparison of the propagation times in Table II with the experimental value for a chain of five balls (0.434 ms) shows, that $r = 1.5$ provides very good agreement between the calculated value and the experimental values.

## V. SOLUTION

There now seems to be a dilemma. The measurement of the propagation time tells us that the interaction is correctly described by Hertz's law. However, the computer simulation with this exponent yields a collision process in which dispersion is not at all negligible. On the other hand, direct visual observation of the ball chain shows that the system is dispersion free to a high degree: one should expect that, if there is a slight dispersion in one collision sequence, this dispersion will be amplified when the last ball swings back, thus provoking a second collision sequence backward through the chain and, similarly, as the first ball swings forward again, etc. After a small number of back and forth collisions, the positions of the balls should be in complete disorder, if the chain obeys Hertz's law.

The reason why it does not, can be discovered by more carefully observing the chain immediately after the first impact. In fact, the balls, which are generally described as being at rest are not at rest at all. They are moving, just as the computer simulation with $r = 1.5$ predicts and not because of any inherent experimental imperfections, as might have been supposed. The apparent dilemma has the following explanation: When, after the first collision sequence, the last ball swings back, the "resting" chain of balls is now in a different state that it was initially. When the last ball swings back, the other balls are not only moving, each is now separated from its neighbors by a small distance. Therefore, the second collision sequence is a dispersion-free propagation for the trivial reason already described in Sec. I. As the position versus time curve of Fig. 4 shows, a separation of only 40 $\mu$m is sufficient for dispersion-free propagation through the chain. Thus the deviation from a perfect dispersion-free system during the first collision sequence creates a perfectly dispersion-free collision system for all the later sequences.

## VI. SUMMARY

A chain of elastic balls suspended in a row and touching one another can be described as a system of mass points interacting by springs of a special type. The springs represent the material of the balls in the immediate neighborhood of the region of contact.

The exponent of the force versus displacement law of these springs is 1.5 in agreement with a theory of H. Hertz. The computer simulation of a ball chain using a force law with an exponent of 1.5 shows that the dispersion is reduced with respect to Hooke's law but not entirely removed. Careful observation of an experimental chain of balls shows that, after collision, the individual balls assumed to be at rest within the chain are actually no longer at rest. This is also in agreement with a computer simulation of the collision process. As a consequence of the slight dispersion within the initial chain, the balls no longer touch after the first collision sequence. This is the reason why, in all the following collision sequences, the chain is a perfectly dispersion-free system.

[1]F. Herrmann and P. Schmälzle, Am. J. Phys. **49**, 761 (1981).
[2]H. Hertz, J. Reine Angew. Math. **92**, 156–171 (1881); reprinted in: H. Hertz, *Gesammelte Werke* (Leipzig, 1895), Bd. 1, pp. 155–173. See also L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics* (Pergamon, London, 1959), Vol. 7 (Theory of Elasticity), pp. 30–36; or A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover, New York, 1927), pp. 193–200.
[3]L. D. Landau and E. M. Lifshitz, Ref. 2, p. 35, Eq. (9.14).