

Analogy between mechanics and electricity

F Herrmann and G B Schmid

Institut für Didaktik der Physik, Universität Karlsruhe, Kaiserstrasse 12, 7500 Karlsruhe 1, West Germany

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Abstract The dissipative transport of energy is described in the momentum current picture. This picture provides a local-causes approach to mechanics whereby forces are considered as momentum currents. In this approach, friction, i.e. mechanical heat production, appears when a momentum current flows between two bodies of different velocities. The treatment of the transport and dissipation of energy follows the same rules in mechanics as in electricity. An ‘Ohm’s Law of momentum currents’ is introduced in analogy to Ohm’s Law in electricity. Newton’s Third Law reduces to a simple statement about momentum conservation.

Zusammenfassung Der dissipative Transport von Energie wird im Impulsstrombild beschrieben. Dieses Bild, in dem Kräfte als Impulsströme aufgefaßt werden, beschreibt die Mechanik nahewirkungstheoretisch. Reibung, d.h. mechanische Wärmeproduktion, tritt auf, wenn ein Impulsstrom zwischen zwei sich mit verschiedener Geschwindigkeit bewegenden Körpern fließt. Die Behandlung von Energietransport und dissipation geschieht in Mechanik und Elektrizitätslehre nach denselben Regeln. Ein ‘Ohmsches Gesetz für Impulsströme’ wird eingeführt, in Analogie zum Ohmschen Gesetz für elektrische Ströme. Das dritte Newtonsche Gesetz erscheint als eine einfache Aussage über die Impulserhaltung

1. Introduction

Figures 1(a) and 1(b) illustrate two simple dissipative processes. In both cases, energy is flowing from one location to another: in the first case, from the man to the underside of the crate; in the second case, from the battery to the light bulb. These processes are typically described as follows:

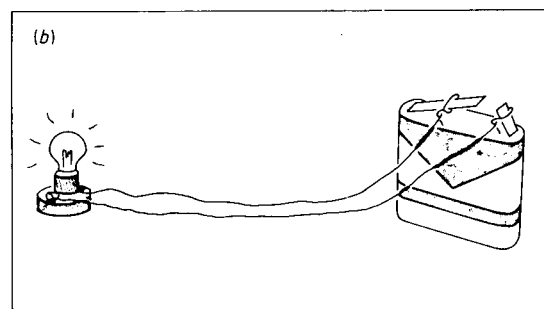
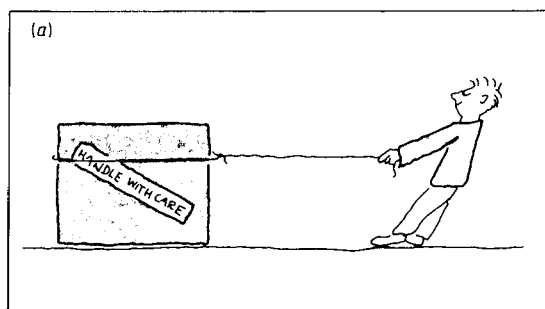
(i) The work done per unit time (=power) to keep the crate moving at a constant velocity v is given by $P = v \cdot F$ where F is the force exerted by

the man on the crate.

(ii) The energy per unit time (=power) dissipated in the light bulb is given by $P = UI_0$ where U is the potential drop maintained by the battery across the light bulb and I_0 is the charge current flowing through the circuit.

Although these processes are closely related physically, their descriptions use quite different concepts: ‘work done’ and ‘force’, in the first case,

Figure 1 Two examples illustrating the flow of energy from one location, an energy source, to another location, an energy receiver, where the energy is dissipated by: (a) a man pulling a crate at a constant speed; (b) a battery operating a light bulb at a constant voltage.



'energy' and 'charge current' in the second. This different usage is not a logical consequence of the dissimilarities between the two processes. Rather, it results from the fact that mechanical processes are described in terms of concepts which are, roughly speaking, one hundred years older than the concepts used to describe electrical processes. For example, a statement like 'X exerts a force on Y' makes no reference to any medium connecting X and Y which could make the exertion of the force possible. This results in an action-at-a-distance picture for cases in which no medium is visible (as, for instance, in gravitation). As another example, a statement like 'a force does work' obviously stems from a time when energy was not yet recognised as a physical quantity of its own right.

In this paper we propose a description of energy transport and dissipation in mechanical processes which takes advantage of the fact that forces are identical to momentum currents (Herrmann 1979, DiSessa 1980) and, accordingly, Newton's second law is identical to the continuity equation for momentum (Herrmann and Schmid 1984).

This description of mechanical processes emphasises a particular analogy between mechanical and electrical processes by which every mechanical quantity corresponds to an electrical quantity: momentum \mathbf{p} to electric charge Q , momentum current \mathbf{I}_p (=force) to electric charge current I_Q and velocity \mathbf{v} to electric potential ϕ (Falk and Ruppel 1976). Furthermore, every relation between mechanical quantities corresponds to a relation of the same mathematical form between the analogous electrical quantities.

To be sure, this analogy, characterised by the correspondence between \mathbf{p} and Q , is mathematically no less justified than other more familiar analogies (Olsen 1958, MacFarlane 1964) (as, for instance, those which associate Q with position \mathbf{r}). However, the analogy considered here has an important additional advantage: it evokes a physical picture which comprises mechanical as well as electrical processes. This is due to the fact that momentum as well as electric charge are 'substance-like' quantities (Falk *et al* 1983): Both momentum ('amount of motion') and electric charge ('amount of electricity') are distributed in space, i.e. have densities, and can flow from one region of space to another, i.e. have current densities. In addition, the corresponding quantities v (velocity) and ϕ (electric potential) are 'energy-conjugates': both relate energy changes dE of a system to momentum changes $d\mathbf{p}$ or charge changes dQ , respectively, according to Falk and Ruppel (1976).

$$dE = \mathbf{v} \cdot d\mathbf{p} + \phi dQ.$$

A correspondence between the scalar Q and the vector \mathbf{p} actually implies a set of three correspondences, namely, between the scalar Q on the one

hand and each of the three components of \mathbf{p} on the other hand (note that momentum conservation is valid for each component separately). However, we restrict our considerations in this paper, for simplicity, to the flow of only one of the three components of momentum. This component can then be treated like a scalar quantity (for a fixed choice of coordinates). This not only makes the analogy to electricity theory particularly obvious; it also shows that the momentum current for each component of the momentum forms a circuit in three dimensions.

For mechanical processes more complicated than those considered here, the simultaneous flow of three components of momentum would have to be considered. This means keeping track of two more momentum flow circuits. If the distribution of momentum currents in the momentum transmitting medium is desired, the entire momentum current density tensor (=negative stress tensor) must be taken into account (Herrmann and Schmid 1984, Landau and Lifschitz 1959).

This paper is organised as follows: section 2 deals with the necessity of distinguishing between two different types of momentum currents. Accordingly, the decomposition of momentum currents is discussed in terms of a similar decomposition of charge currents in electricity. An 'Ohm's law of momentum currents' for mechanical processes with friction is then introduced in analogy to the well-known Ohm's law for charge flow with energy dissipation. In section 3, Newton's third law is formulated in the momentum current picture and the advantages of this formulation are discussed. The considerations in sections 2 and 3 facilitate the description of energy transport with momentum currents in section 4. Section 5 contains conclusions.

2. Types of current

Figure 2 shows two possible ways to give a body momentum: In the first case, momentum flows through a rod to the wagon; in the second case, momentum flows with the water from a hose to the wagon. Consider a cross-sectional area cut through the rod in the first example and through the water stream in the second. In the first example, one does not traditionally speak of a momentum current crossing this area. One rather says that the rod to the left of the area exerts a force on the rod to the right. In the second example, it would be incorrect to say that the water to the left of the area exerts a force on the water to the right.

In the momentum current picture, we speak of a momentum current in both of the above cases. In the first case, we will call this a *conductive* momentum current: no momentum density is associated with the momentum current density. In the second case, we will call this a *convective* momentum current: a non-zero momentum density is associated

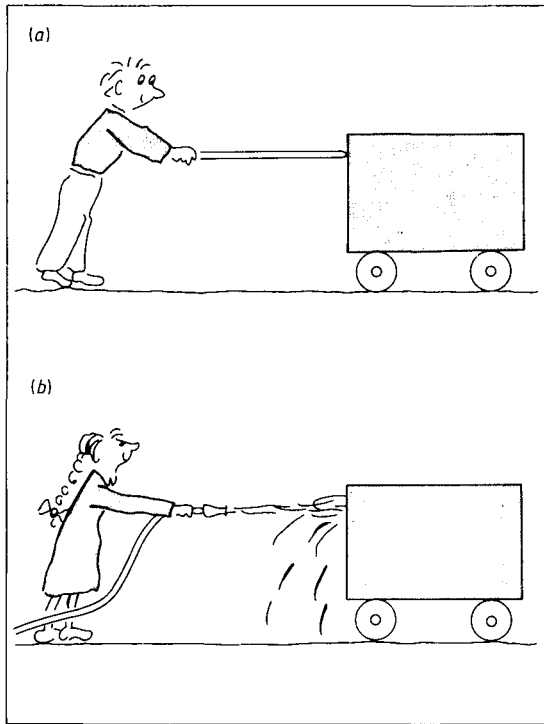


Figure 2 Two possible ways to give a body momentum: (a), momentum flows through a rod to a wagon; (b), momentum flows with a stream of water to a wagon.

with the momentum current density. Accordingly, forces are identical to conductive momentum currents.

Conductive and convective momentum currents can be distinguished by their transformation properties: Convective momentum currents can be transformed away by a suitable choice of reference frame, conductive momentum currents cannot. This is similar to the distinction between conductive and convective electric currents. The momentum current illustrated in figure 2(a) is similar to an electric current flowing in, say, a metal; the momentum current illustrated in figure 2(b) is similar to, say, the flow of electrons in a vacuum tube. The latter electric current can be transformed away, the former not.

The above analogy goes even further. Just as there are electric super currents, i.e. electric currents which flow without energy dissipation, there are also momentum super currents, i.e. momentum currents which flow without energy dissipation. Typical examples of momentum super currents are the conductive momentum currents flowing through elastic media (Herrmann and Schmid 1984) or through fields. Whereas electric super currents are the exception, momentum super currents are the rule. Of course, at high enough temperatures, all

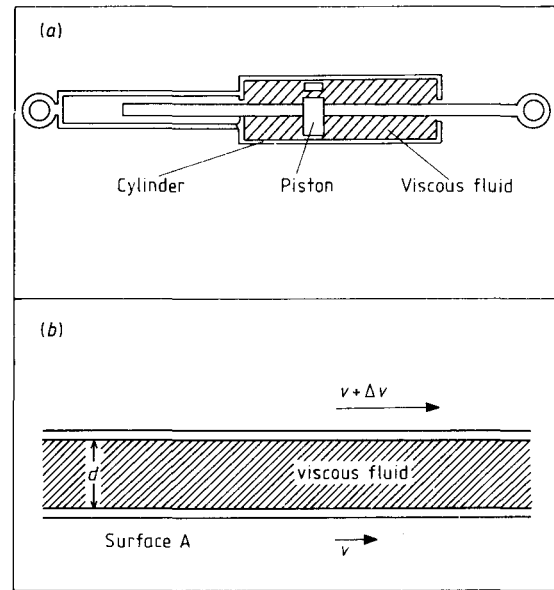


Figure 3 (a), In the momentum current picture, a dash pot is a momentum resistor. (b), Two surfaces separated by a viscous fluid are sliding past one another with relative speed Δv . The area of contact between the surfaces and the fluid is designated A and the thickness of the fluid is d .

momentum super currents become dissipative, namely, when the momentum conducting medium begins to flow or to melt. Dissipation is always present in a momentum current circuit where, traditionally speaking, frictional or damping effects occur during the exertion of a force on a body (figures 3(a) and 3(b)). Just as one defines an electric resistance R_O as

$$R_O = \Delta\phi / I_O \quad (1)$$

a momentum resistance R_p can be defined as

$$R_p = \Delta v / I_p \quad (2)$$

In equation (1), $\Delta\phi$ is the potential difference across the resistive medium through which the electric current I_O is flowing and R_O is the associated resistance to this flow. In equation (2), Δv is the magnitude of the velocity difference across the resistive medium through which the momentum current of magnitude I_p is flowing and R_p is the associated resistance to this flow.

Just as R_O can often be expressed in terms of the length d , cross-sectional area A and electric conductivity σ of a charge conducting channel

$$R_O = (1/\sigma)(d/A) \quad (3)$$

the momentum resistance R_p can often be expres-

sed in terms of the length d , cross-sectional area A and viscosity η of a momentum conducting channel (recall figure 3(b))

$$R_p = (1/\eta)(d/A). \quad (4)$$

Equation (4) can be seen to follow directly from equation (2) with the help of the well-known relation $I_p (= F) = \eta(A/d)\Delta v$ for the momentum current (force) flowing through (acting on) the underside of a body sliding along a viscous layer (Feynman *et al* 1964).

3. Newton's third law

In the momentum current picture, Newton's first law states that the amount of momentum contained within a body does not change as long as the net flow of momentum into or out of the body is zero; Newton's second law simply expresses the continuity equation for momentum (Herrmann 1979, Herrmann and Schmid 1984). Newton's third law, in the usual force picture, can be stated as (Purcell 1973):

Whenever two bodies interact, the force \mathbf{F}_{21} that body 1 exerts on body 2 is equal and opposite to the force \mathbf{F}_{12} that body 2 exerts on body 1:

$$\mathbf{F}_{12} = -\mathbf{F}_{21}.$$

In the momentum current picture, Newton's third law reads:

Whenever a momentum current is flowing between two bodies, the momentum current \mathbf{I}_p (enter 1), entering body 1 is equal to the momentum current \mathbf{I}_p (leave 2) leaving body 2;

$$\mathbf{I}_p(\text{enter 1}) = \mathbf{I}_p(\text{leave 2}).$$

The formulation of an analogous law for the flow of electric charge between two bodies is so trivial as to be left out of most lectures or textbooks on electricity theory.

To understand how the momentum current formulation of Newton's Third Law follows from the force formulation, consider the latter translated literally into the momentum current picture. Whenever a momentum current flows between two bodies, the momentum current \mathbf{I}_p (enter 2) entering body 2 is equal and opposite to the momentum current \mathbf{I}_p (enter 1) entering body 1:

$$\mathbf{I}_p(\text{enter 1}) = -\mathbf{I}_p(\text{enter 2}).$$

This formulation is unnecessarily awkward due to the introduction of a negative current. However, a negative current entering a body is equivalent to a positive current leaving a body. Accordingly, this literal translation can be rewritten more clearly in the form suggested above. Often the channel (which could, for example, be a magnetic field) connecting two bodies accumulates no momentum, i.e. is taken to be massless. Then Newton's third

law states that the rate of momentum flow into the channel at one end is equal to the rate of momentum flow out of the channel at the other end. Considering that momentum is a conserved quantity (Newton's second law), this is actually a triviality.

4. Energy transport with momentum currents

In view of the considerations in sections 2 and 3, we now return to the correspondence between the processes sketched in figures 1(a) and 1(b). Just as a charge current is flowing in a closed loop in figure 1(b), a momentum current is flowing in a closed loop in figure 1(a), namely, from the muscles in the man's arms into the rope, through the rope into the crate, through the crate and out through the bottom of the crate into the earth, through the earth and, finally, back again into the man. In the example of figure 1(b), there is an electric potential difference $\Delta\phi$ between the two channels (the wires) connecting the energy source (the battery) to the energy receiver (the light bulb). The energy source in this case maintains $\Delta\phi$ at a constant value. In the example of figure 1(a), there is a velocity difference Δv between the two channels (rope and earth) connecting the energy source (the man's muscles) to the energy receiver (the underside of the crate). The energy source in this case maintains Δv at a constant value. If one of the charge conducting channels in figure 1(b) is grounded under the convention that $\phi_{\text{earth}} = 0$, this channel will also be at zero potential. In figure 1(a) one of the momentum conducting channels is already 'grounded'. Under the convention that $v_{\text{earth}} = 0$, this channel is at zero velocity.

Figure 4 shows how the momentum flow in the two channels can be demonstrated with springs in a

Figure 4 Sketch of a possible lecture demonstration to show the channels through which momentum is flowing from the muscles of a man to a crate and back again. Taking the direction from left to right to be positive, the expanded spring in the rope evidences the flow of positive momentum in the negative direction whereas the compressed spring between the boards evidences the flow of positive momentum in the positive direction. The rollers under the boards insulate the momentum circuit from the earth.

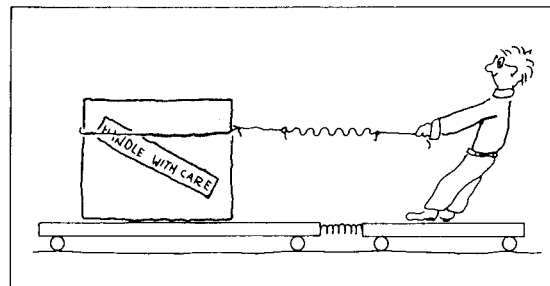


Table 1 Summary of the analogy between mechanics and electricity.

Electricity	Quantities	Mechanics
Electric charge Q		Momentum p_i ; $i = 1, 2, 3$
Charge current I_O		Moment current I_p ; $i = 1, 2, 3$
Electric potential ϕ		Velocity v_i ; $i = 1, 2, 3$
Charge conductivity σ		Viscosity η
Electric resistance R_O		Mechanical resistance R_p
Charge capacity C		Momentum capacity m
Electrical inductance L		Mechanical inductance $1/k$ (= 'reciprocal of spring constant k ')
	Relations	
	$dQ/dt = 0$ for isolated bodies	$d\mathbf{p}/dt = 0$ for isolated bodies
	$dQ/dt + I_O = 0$	$d\mathbf{p}/dt + \mathbf{I}_p = 0$
	$I_O(1) = -I_O(2)$ for the flow of charge between two bodies	$\mathbf{I}_p(1) = -\mathbf{I}_p(2)$ for the flow of momentum between two bodies
	$R_O = \Delta\phi/I_O$	$R_p = \Delta v/I_p$
	$N\phi_B = LI_O$; N = number of turns in a coil	$kx = I_p$; x = displacement from equilibrium
	ϕ_B = magnetic flux set up in each turn	position of the end of a spring
	$P = \Delta\phi I_O$	$P = \Delta v \cdot \mathbf{I}_p$
	$Q = C\Delta\phi$	$\mathbf{p} = m\Delta v$

lecture hall. A spring lying along the x axis is compressed or expanded when positive x momentum is flowing through it in the positive or negative x direction, respectively (Herrmann and Schmid 1984). The rollers under the boards insulate the momentum circuit from the earth: rollers and wheels are momentum insulators. Just as the flow of charge in the current loop in figure 1(b) is simultaneous to the flow of energy from an energy source (the battery) to an energy receiver (the light bulb), the flow of momentum in the current loop in figure 1(a) is simultaneous to the flow of energy from an energy source (the man's muscles) to an energy receiver (the underside of the crate). In both cases, energy is dissipated at the energy receiver. The intensity P of the energy current flowing from source to receiver, i.e. the transmitted power is given by

$$P = \Delta\phi I_O \quad (5)$$

$$P = \Delta v \cdot \mathbf{I}_p \quad (6)$$

for figure 1(b) and 1(a) respectively.

One apparent difference between the situations illustrated in figures 1(a) and 1(b) is that, whereas the crate in figure 1(a) has net momentum, the light bulb in figure 1(b) has no net charge. This difference, however, can be easily avoided, for example, by replacing the light bulb in figure 1(b) with a capacitor and ohmic resistor in parallel. The relation

$$Q = CU \quad (7)$$

where C is the charge capacity of the capacitor corresponds to the relation

$$\mathbf{p} = mv. \quad (8)$$

This comparison shows that the mass m of a body can be understood as the momentum capacity of the body.

5. Conclusions

The momentum current picture of mechanics has several advantages over the traditional picture in so far as the former:

(i) Stresses a local-causes point of view in the description of mechanical processes.

(ii) Corresponds structurally to the traditional presentation of electricity (see table 1). As will be shown elsewhere, this correspondence can be extended to include the treatment of processes in rotational dynamics, thermodynamics and chemistry as well when these are based upon the flow of angular momentum, entropy and amount of substance, respectively. Indeed, the approach to physics introduced here is only part of a generalised dynamics (Falk 1968, Schmid 1984) valid throughout all of physics.

(iii) Is easy to use. In particular, the confusion typical of the traditional interpretation of Newton's third law is avoided altogether.

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