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An analogy between information and energy

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Abstract The total entropy of an information storage system can be decomposed into independent terms, i.e. into functions which have no independent variables in common. One of these terms represents the information (= entropy) in which the user of a computer is interested. This decomposition corresponds to a break-up of the entire system into non-interacting subsystems and is analogous to the decomposition of the total energy of a system into independent terms commonly referred to as energy forms. In both decompositions, the term usually of interest is many orders of magnitude smaller than the rest.

Zusammenfassung Es wird gezeigt, daß die Gesamtentropie eines Informationsspeichers in unabhängige Terme zerlegt werden kann, d.h. in Funktionen, die keine gemeinsamen unabhängigen Variablen haben. Einer dieser Terme stellt die Information (= Entropie) dar, welche für den Benutzer des Computers von Interesse ist. Diese Zerlegung entspricht einer Aufspaltung des Systems in Teilsysteme, die nicht miteinander wechselwirken. Sie ist analog zu der Zerlegung der Gesamtenergie eines Systems in Energieformen. In beiden Zerlegungen ist der interessante Term um viele Größenordnungen kleiner als der Rest.

1. Introduction

The entropy S of a physical system can be expressed (Reif 1965) as a function of the probabilities $p(i)$ of the microstates of the system:

$$S = -k \sum_i p(i) \ln p(i) \qquad (1)$$

where k is Boltzmann's constant. The macrostate of the system is determined by the distribution $\{p(i)\}$ of the probabilities $p(i)$.

I, the Shannon measure of the information of a system, is given by a relation (Shannon and Weaver 1949) having the same mathematical structure as (1):

$$I = -f \sum_i p(i) \ln p(i). \qquad (2)$$

Here I is the (amount of information)/symbol generated by a source, $p(i)$ is the probability that symbol i will be generated and f is a constant of proportionality. The usual unit of I is the bit. In these units, the constant f has the value (1/ln 2) bit.

Equation (2) can be applied to a thermodynamic system and interpreted as the amount of information which an observer would have about the system if he were to receive a signal telling him what microstate the system was in. For this reason, it is customary to call the physical entropy defined by (2) the 'amount of missing information' (of the observer) about the microstate of the system (Rothstein 1951, 1952a, b, Jaynes 1957). In spite of the ambiguity of the terms—entropy and missing information—it has long been proven that entropy as defined by (1) and Shannon's measure of information (2) are physically identical (Tribus and McIrvine 1971):

$$S = (k/f)I. \qquad (3)$$

The equivalence (3) shows that

$$1 \text{ bit} = 0.96 \times 10^{-23}\ \text{J K}^{-1} \simeq 10^{-23}\ \text{J K}^{-1}. \qquad (4)$$

In view of this equivalence, consider now an information storage device, for example that of a computer. The entropy (= information) of this macroscopic system is of the order of 1 J K^{-1} which, according to (4), equals 10^{23} bit.

However, if we ask a designer or user of the computer how much information is contained within the computer's memory, we may expect to get an answer like '1 Mbit', i.e. 10^{17} times smaller than the actual value.

What is the reason for this discrepancy? How can the minute amount of information important to the user of a computer be distinguished from the very much larger rest? It is the purpose of this paper to show the answers which thermodynamics provides to these questions. In §§ 2 and 3 it will be shown that the total entropy S of an information storage device can be decomposed into two parts:

$$S = S_{nt} + S_t \tag{5}$$

where S_{nt}, the non-thermal entropy, is the stored information and S_t, the thermal entropy, is a much larger remainder. In §§ 4 and 5 it will be shown that (5) is analogous to the more familiar decomposition of the total energy stored in a system into different 'energy forms'.

2. Specification of the state of a 1-bit computer storage element

Consider a physical system which can be brought into either one of two possible states for the purpose of information storage, say, a magnet which can be magnetised along either one of two possible directions. The magnetisation m along these directions is either of the two values $+m_0$ or $-m_0$.

Now assume the system finds itself (with probability 1) in a state with $m = +m_0$. Then the probability distribution $\{p(i, m_0)\}$, $i = 1, 2, \ldots, N$ of the N microstates (i) of the system is fully specified. If the system is subsequently brought into a state with $m = -m_0$ while holding the values of the temperature and all other macroscopic variables fixed, the resulting probability distribution $\{p(i, -m_0)\}$, $i = 1, 2, \ldots, N$ is the same as above.

Next consider a Shannon information source which can generate either one of two symbols labelled by the index $j = 1$ or $j = 2$. The probability of generating either symbol is designated $p^m(j)$. Each time the source emits one symbol, the magnetisation of the above-mentioned binary storage system is switched accordingly.

For our binary storage system, the storage of a value means that the number of its microstates is doubled from N to $2N$. There are now two classes, each with N microstates. For the class $j = 1$, m has the value $+m_0$ and for the class $j = 2$, m has the value $-m_0$. Every microstate belonging to the one class corresponds to a microstate of the other and vice versa.

A microstate i belonging to one class differs from the corresponding microstate i belonging to the other class only by its value of m. Every microstate is characterised by two numbers i and j and its probability $p(i,j)$ of occurrence can be expressed as a product of the independent probabilities $p^0(i)$ and $p^m(j)$:

$$p(i, j) = p^0(i) p^m(j) \tag{6}$$

with

$$\sum_{i=1}^{N} p^0(i) = 1 \tag{7}$$

and

$$\sum_{j=1}^{2} p^m(j) = 1. \tag{8}$$

3. The decomposition of the entropy of an information storage system

Inserting (6) into (1), summing over i and j, and simplifying the resulting expression using (7) and (8), we get:

$$S = -k \sum_{i=1}^{N} p^0(i) \ln p^0(i) - k \sum_{j=1}^{2} p^m(j) \ln p^m(j). \tag{9}$$

The first sum on the right-hand side of (9) is the entropy which the storage device would have if the probability of either possible value of m_0 were 1. We designate this the *thermal* part of the total entropy with the symbol S_t. The second sum on the right-hand side of (9) is that part of the total entropy in which a communication theorist is most interested. We designate this *non-thermal* part of the total entropy by the symbol S_{nt}. Accordingly,

$$S(p^0(i), p^m(j)) = S_t(p^0(i)) + S_{nt}(p^m(j)). \tag{10}$$

Equation (10) represents a decomposition of the total entropy of a system into two independent terms: the first term depends only upon the $p^0(i)$; the second term only upon the $p^m(j)$. This corresponds to a decomposition of the system into two noninteracting subsystems.

The independence of the $p^0(i)$ from the $p^m(j)$ is equivalent to the fact that no thermal equilibrium exists between the corresponding subsystems. Indeed, this fact allowed us to use the subsystem associated with the $p^m(j)$ as an information storage device in the first place. The subsystem characterised by the $p^0(i)$, on the other hand, will in general be in thermal equilibrium with the environment of the entire system. S_t is generally a function of the temperature T alone whereas S_{nt} is a function of the average value $\langle m \rangle$ of the magnetic moment

$$\begin{aligned} \langle m \rangle &= p^m(1)m_0 + p^m(2)(-m_0) \\ &= p^m(1)m_0 + (1 - p^m(1))(-m_0) \\ &= (2p^m(1) - 1)m_0. \end{aligned} \tag{11}$$

$S_{nt} = S_{nt}(\langle m \rangle)$ because, given $\langle m \rangle$, $p^m(1)$ (and consequently $p^m(2) = 1 - p^m(1)$) can be determined from the above equation thus specifying the distribution corresponding to S_{nt}. Accordingly, instead of (10), we can rewrite the total entropy S of the system in terms of the 'macroscopic' variables T and $\langle m \rangle$ as

$$S(T, \langle m \rangle) = S_t(T) + S_{nt}(\langle m \rangle). \tag{12}$$

4. The decomposition of the energy of a system

The manner in which the total entropy of a system is broken up into two parts has been copied from the way (Falk and Ruppel 1976, Falk *et al* 1983) a similar decomposition is carried out for the energy. Consider the energy E of an arbitrary system as a function of its independent variables $\{X_i\}$:

$$E = E(X_1, X_2, \ldots). \tag{13}$$

For certain systems, this function breaks up into a series of terms, each depending upon variables not belonging to the other terms of the series.

To see this explicitly, consider a moving capacitor. Let the momentum of this capacitor be p, its charge Q, its mass M and its capacity C. The the total energy E can be written as

$$E = E(p, Q) = E_0 + E_1(p) + E_2(Q) = E_0 + p^2/2M + Q^2/2C. \quad (14)$$

The individual terms of such a decomposition often have separate names. In the above case, E_0 is called the 'rest energy', $E_1(p)$ the 'kinetic energy' and $E_2(Q)$ the 'electric field energy' of the capacitor. Of course, such a decomposition is not always possible. For example, consider the expression for the energy of an ideal gas as a function of entropy, volume and amount of substance.

5. Comparison of the decomposition of the entropy and energy of a system

Some systems can be broken up in such a way that their total entropy decomposes into a relation of the form (12) (recall § 3). Such systems are useful for data storage. Similarly, some systems can be broken up in such a way that their total energy decomposes into a relation of the form (14) analogous to (12) (recall § 4). It will be shown in this section that this analogy between entropy and energy also includes the relative orders of magnitude of the individual terms appearing in (12) and (14), respectively, for typical technical systems.

The value of S_t in (12) is of the order of 1 J K^{-1} for a computer memory. The value of S_{nt} is typically $1 \text{ Mbit} = 10^{-17} \text{ J K}^{-1}$, i.e. very much less than the value of S_t. Thus, as far as one is concerned about the total entropy S of such a system, it is irrelevant whether one takes the value of S itself or S_t.

The value of E_0 in (14) is of the order of 10^{15} J for, say, a capacitor weighing 10 g and having a capacitance of $1\ \mu$F. If this capacitor is moving at 1 m s^{-1} and is charged up to 100 V, the values of E_1 and E_2 in (14) are each 0.005 J. Thus, as far as one is concerned about the total energy E of such a system, it is irrelevant whether one takes the value of E itself or E_0.

The values of S_{nt} and E_1 or E_2 are much smaller than the values of S_t and E_0, respectively, and are even much smaller than the natural fluctuations in time of S_t and E_0. For example, a change in temperature of a computer memory by only 0.01 K effects a change in the value of S_t of the order of 10^{-3} J K^{-1}, i.e. 14 orders of magnitude greater than the value of S_{nt}. In the same way, the change in the total energy of an accelerating car results more from, say, the wear on its tyres (according to $\Delta E = \Delta mc^2$), than from the gain in its kinetic energy.

We now return to the question raised in the introduction: 'How can the minute amount of information important to the user of a computer be distinguished from the very much larger rest?' This question is analogous to the question: 'How can the minute amount of kinetic energy important to the driver of a car be distinguished from the very much larger amount of rest energy of the car?'. The answers to these questions follow from equations (12) and (14). To determine the value of S_{nt} or E_1, one does not determine the value of an entropy or an energy itself but, rather, only the values of other variables upon which only S_{nt} or E_1, respectively, depend: in the above case of a computer storage element, only the average value of the magnetic moment m and in the case of the car, only the value of the momentum p. From these, the values of S_{nt} and E_1, respectively, are calculated.

6. Conclusions

The information important to the user of a computer is part of the very much larger physical entropy of an information storage system. This 'non-thermal' part of the entropy can be determined because its value depends upon variables from which the rest of the entropy, i.e. the 'thermal' part, is independent. The decomposition of the total entropy of a system into independent terms, i.e. into respective functions of independent variables, corresponds to a break-up of the entire system into non-interacting subsystems. This decomposition is analogous to the decomposition of the total energy of a system into independent terms. In both cases, the term usually of interest is many orders of magnitude smaller than the rest.

References

Falk G, Herrmann F and Schmid G B 1983 *Am. J. Phys.* **51** 1074
Falk G and Ruppel W 1976 *Energie und Entropie* (Berlin: Springer) p 137
Jaynes E T 1957 *Phys. Rev.* **106** 620
Reif F 1965 *Fundamentals of Statistical and Thermal Physics* (New York: McGraw-Hill) p 219
Rothstein J 1951 *Science* **114** 171
—— 1952a *Phys. Rev.* **85** 135
—— 1952b *J. Appl. Phys.* **23** 1281
Shannon C E and Weaver W 1949 *The Mathematical Theory of Communication* (Chicago: University of Illinois Press)
Tribus M and McIrvine E C 1971 *Sci. Am.* September 179